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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4 **(82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol 22, No. 9A September, 1961 Reviews 7913–8537

GENERAL

See also 8086, 8392, 8393, B8782.

7913:

★American Mathematical Society Translations. Series 2, Vol. 16. American Mathematical Society, Providence, R.I., 1960. v + 479 pp. $5.00.

Translations of five papers [#8133, 8161, 8162, 8366, 8367] and of 35 brief communications in Uspehi Mat. Nauk [#8077, 8163, 8164, 8223, 8231, 8255, 8264, 8278, 8368–8390, 8437, 8952, 9112, 9113].

7914:

Wissenschaftliche Jahrestagung der Gesellschaft für Angewandte Mathematik und Mechanik in Freiberg (Sachsen) (19–23 April 1960). Z. Angew. Math. Mech. **40** (1960), T1–T155.

This issue of ZAMM is devoted to the lectures presented at the annual meeting of the Gesellschaft. There are 56 summaries, of varying length, grouped under the headings of (a) Applied mathematics, (b) Numerical techniques, (c) Probability theory and statistics, (d) Mechanics, (e) Elasticity theory, (f) Fluid mechanics.

7915:

Sauer, Robert. ★Ingenieur-Mathematik. Erster Band. Differential- und Integralrechnung. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. viii + 304 pp. DM 24.00.

This first volume of a two-volume undergraduate textbook deals with differential and integral calculus, including an introduction to vector analysis, linear algebra and two- and three-dimensional analytical geometry. The second volume is to treat differential equations, theory of functions, the integral theorems of vector analysis and Fourier series.

Mathematical rigour is achieved in the context of a book addressed to engineers by leaving the detailed proofs of many theorems to the reader, without glossing over them. Much space is given to numerical methods, both in order to elucidate the theory and to present useful procedures. The book is excellently produced and beautifully printed; there are many fine drawings to illustrate the analytical concepts. *W. Freiberger* (Providence, R.I.)

7916:

Wylie, C. R., Jr. ★Advanced engineering mathematics. 2nd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. xi + 696 pp. $9.00.

This is the second completely rewritten edition of the textbook first published in 1951 [McGraw-Hill, New York]. It begins with a chapter on determinants and matrices. This is followed by three chapters on differential equations and one on finite differences, including curve-fitting, data smoothing and orthogonal polynomials. Applications to mechanical and electrical systems follow, exploiting the analogy between the two fields. Next, Fourier series and integrals and Laplace transforms are introduced, followed by separable partial differential equations, Bessel and Legendre functions, vector analysis and, lastly, four chapters on complex function theory. There are interspersed among the chapters several topics from numerical analysis, and a great many illustrative examples and exercises. The material is chosen to include all applicable mathematics an engineering student must know to master his engineering courses.

7917:

Puckett, Allen E.; Ramo, Simon (Editors). ★Guided missile engineering. University of California Engineering Extension Series. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1959. viii + 497 pp. $10.00.

Each of the eighteen authors summarizes the concepts and techniques underlying his field of specialization. The discussion ranges from aerodynamics through propulsion, guidance and navigation, radio and radar, computers and simulators, to integrated system design.

One might well hope that our missile designers are better and more widely informed than one is led to believe by the elementary nature of this survey.

J. R. Ward (Providence, R.I.)

7918:

Steinhaus, Hugo. ★Kaleidoskop der Mathematik. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 319 pp. DM 19.80.

Approximately the same in content as the recent American edition [Oxford Univ. Press, New York, 1960; MR **22** #5541]. The drawings are adapted to the European reader and many are in color.

7919:

Menninger, Karl. ★Zwischen Raum und Zahl: Mathematische Streifzüge. Ullstein Taschenbücher-Verlag, GmbH., Frankfurt/M., 1960. 222 pp. DM 2.20.

A popularization, at an elementary level and on the theme of geometry and arithmetical geometry, consisting of a profusion of interesting facts, demonstrations, quotations, poems and elegant line drawings, bound together by an appreciation of the ubiquity and order of mathematics.

7920:

Cutler, Ann; McShane, Rudolph. ★The Trachtenberg speed system of basic mathematics. Doubleday & Co., Inc., Garden City, N.Y., 1960. 270 pp. $4.95.

A number of clever devices for performing mental arithmetic, allegedly invented by Trachtenberg while a prisoner in German concentration camps, are here accompanied by bombastic claims, for example: "... mathematical experts believe that within the next decade the Trachtenberg system will have as far reaching an effect on education and science as the introduction of shorthand did on business." The book is of interest to mathematicians concerned with the teaching of elementary arithmetic and to arithmetic teachers in search of interesting variations. However, Trachtenberg's tricks, suitable as they may be for maintaining one's sanity in a concentration camp, hide the true nature of the arithmetical operations even more effectively than the usual algorithms. They are important only if our purpose is to produce a large number of "idiots savants". *Kenneth May* (Northfield, Minn.)

7921:

Wansink, Joh. H. Mathematical olympiads. Euclides (Groningen) **36** (1960/61), 109–117. (Dutch)

An account of the various national competitions.

7922:

Burington, Richard Stevens. The problem of formulating a problem. General considerations. Proc. Amer. Philos. Soc. **104** (1960), 429–443.

An informative and well illustrated account of the general sociological-scientific situation of the applied mathematician today, discussed in the light of an analysis of the development of selected mathematical and physical theories, in a rather clear and balanced fashion. The problem of the technical formulation of a question formulated originally in general terms is shown to be commonly, and especially in the work of the practicing applied mathematician (on whose role this article is particularly authoritative), the really definitive question.

I. E. Segal (Cambridge, Mass.)

7923:

Doob, J. L. Some problems concerning the consistency of mathematical models. Information and decision processes, pp. 27–33. McGraw-Hill, New York, 1960.

Some examples of how a mathematician forms a model of a physical problem. It is emphasized that extremely delicate properties of the model do not necessarily correspond to physical reality although they need to be studied when the self-consistency of the model is under study. Brownian motion is mentioned as an example where it is easy to make too many assumptions and so to arrive at an inconsistent mathematical model.

I. J. Good (Teddington)

7924:

de Blavette, Élie. Études de science musicale. Rev. Sci. **91** (1953), 163–211; **92** (1954), no. 2, 3–14.

The common musical scale traverses incommensurable distances, since $(\frac{3}{2})^n$ cannot be divided by 2^m; in particular, the discrepancy between twelve musical fifths $(=(\frac{3}{2})^{12})$ and seven octaves $(=2^7)$ is known as the Pythagorean Comma. The musical "Just" major triad is said to have its prototype in terms 4, 5, 6 of the harmonic overtone series; but the equally important minor triad allows of no comparable rationalization. The worldwide diversity of musical materials and their demonstrable variation throughout history testify to cultural conditioning, and thus against the claims of a uniquely natural status for such elements as the recent occidental major scale; yet that scale's presumed superiority continues to be sought in the numerical formulation of its acoustic constitution.

These are the problems to which M. de Blavette's exercises in musical numerology are directed. His particular hobby is a gratuitous postulate of a symmetrical inversion or reciprocation principle deemed immanent in musical pitch relations.

Every few months somebody re-discovers that reversing the sequence of tones and semitones of the ordinary major scale from c up to c' produces the descending ancient Greek "Dorian" scale from e down to E. Thereby the former's contravention of older church mode dogmas is ruled legitimate, with the further gratifying implication that a profound unity underlies Western musical history. Robert Waterman Stevens announces the discovery with as inspired an air in 1915 as does Hugo Kauder in 1960 [*Counterpoint*, Macmillan, New York]. Surely informed of neither statement, M. de Blavette presents all those enumerated proofs needed by those denied access to a piano keyboard; but he comes no closer to justifying the undertaking.

Inverse derivation of the minor triad requires more strenuous effort, and correspondingly more tenuous hypotheses. But de Blavette's "renversement du sens" does not rest merely on calculation; it claims proper physical virtue for the minor harmony via a postulated undertone or subharmonic series ("la résonnance inférieure"), conceived as a mirror inversion of the harmonic overtone series usually taken as the source of the major harmony. In this postulate he merely repeats without acknowledgment the "tonicity and phonicity" doctrine of Arthur Von Oettingen [*Harmoniesystem in dualer Entwicklung*, 1866], adopted by Hugo Riemann and others. De Blavette mildly confesses that the literal existence of such an undertone series is "contested"; he ignores the comprehensive and masterful demolition of the concept and its imagined evidence, given by Carl Stumpf [*Beiträge zur Akustik und Musikwissenschaft*, VI, 1911] and since generally accepted.

In further development M. de Blavette hides his proclivity for inversions beneath formulations of the mystic "golden section" φ. In manipulation of "le nombre d'or" he surreptitiously introduces its reciprocal $1/\varphi$. But if the given abstract reciprocal $1/\varphi$ be translated into acoustic reference, it would require that ideal monochord strings vibrate in segments that are multiples of unity in length, which is absurd.

De Blavette otherwise glosses over but eventually admits the inevitable imprecision of scale intonation specifications, wherein his derivations from such fanciful "golden" formulas as φ^5 yield less useful approximations than does the standard $2^{1/12}$ of equal temperament; for example, $\varphi^{36}/2^{24}=1.988\ldots$ instead of the desired octave 2. Hence his plea against the adoption of "valeurs artificiels" would seem to beg the question. Our suspicions are further increased when we find that, whenever convenient for his argument, primordial symmetry suddenly

gives way to an eccentric or ellipsoidal law of asymmetrical "gravitation", of equally arbitrary application.

The urbane and intelligent presentation of M. de Blavette's mathematical exercises may best be appreciated as a relatively harmless waste of time by some appropriate inversion of his project title, such that "Études de Science Musicale" is read more accurately as "Spéculations sur l'Astrologie Pseudo-Acoustique".

N. Cazden (Lexington, Mass.)

HISTORY AND BIOGRAPHY

See also 7934, B8784.

7925:

Gazis, Denos C.; Herman, Robert. Square roots geometry and Archimedes. Scripta Math. **25** (1960), 228–241.

The authors add yet another conjecture on how Archimedes might have arrived at his approximations for square roots which have long puzzled his readers. Especially they are able to obtain his limits $1351/780 > \sqrt{3} > 265/153$ from the easily derivable values $7/4 > \sqrt{3} > 5/3$ by an elementary geometrical construction which is analytically equivalent to the Bailey iteration formula of third order: If b_1 is a first approximation to $\sqrt{a}$, then $b_2 = (3ab_1 + b_1^3)/(a + 3b_1^2)$. (It can also be derived from the continued fraction for $\sqrt{(a^2+b)}$.) Their suggestion of a synthesis method by which Archimedes might have obtained such iteration formulae even for higher roots seems, to the reviewer, to deviate too far from the spirit of Greek mathematics. For a more plausible explanation see J. E. Hofmann, Centaurus **5** (1956), 59–72 [MR **18**, 453].

C. J. Scriba (Toronto)

7926:

Fraenkel, Abraham A. Jewish mathematics & astronomy. Scripta Math. **25** (1960), 33–47.

After indicating the scope of mathematical references in the Talmud and the development of the Jewish luni-solar calendar, and sketching the scientific work of Levi Ben Gershon (ca. 1300), the author describes briefly the careers of the following modern mathematicians: C. G. J. Jacobi, C. S. Slonimski, J. J. Sylvester, F. G. M. Eisenstein, L. Kronecker, Z. H. Schapira, G. Cantor, H. Minkowski, K. Schwarzschild, E. Landau, and Emmy Noether.

E. S. Kennedy (Beirut)

7927:

Hofmann, Joseph Ehrenfried. ★Classical mathematics: A concise history of the classical era in mathematics. Philosophical Library, New York, 1959. 159 pp. $4.75.

Translation by Henrietta O. Midonick of Parts 2 and 3 of the author's *Geschichte der Mathematik* [de Gruyter, Berlin, 1957; MR **19**, 518]; the bibliographical indexes of the latter are omitted. Part 1 [1953; MR **15**, 275] was published in translation as *The history of mathematics* [1957; MR **19**, 107].

7928:

Robinet, André. Le groupe malebranchiste introducteur du Calcul infinitésimal en France. Rev. Hist. Sci. Appl. **13** (1960), 287–308.

An annotated bibliography of documents and manuscripts by Malebranche and others relevant to the introduction of calculus into France c. 1690–1700.

7929:

Stone, Marshall H. Mathematics in continental China, 1949–1960. Amer. Math. Soc. Not. **8** (1961), 209–215.

Reproduced from the report of the symposium of the American Association for the Advancement of Science on "Science in Communist China", Washington, D.C., December 1960.

7930:

Gridgeman, N. T. Geometric probability and the number π. Scripta Math. **25** (1960), 183–195.

A history of the Buffon needle problem, its generalizations, and experimental approximations of π based upon these. The author argues from statistical considerations that the method is quite impractical, e.g., more than 10,000 throws of the needle would be required to establish even the first decimal place with 95% confidence. He suggests that several surprisingly accurate experimental determinations were obtained by such devices as stopping at a point where the frequency happened to be favorable, and he offers his own experiment in which by judiciously choosing dimensions and stopping after two throws he finds π to be 3.143! The article is marred by eccentricities of exposition that should not be allowed to reach the pages of a mathematical journal. These include the careless use of mathematical terms (e.g., "straight line" for "straight line segment"), the introduction of useless ad hoc jargon (e.g., "quantal methods", "enumerative methods", "randomicity"), and statements whose meaning is doubtful (e.g., "... we treat the square-root of a variance as a mathematical differential ...").

Kenneth May (Northfield, Minn.)

7931:

Steinhaus, Hugo. Stefan Banach, 1892–1945. Rev. Polish Acad. Sci. **5** (1960), no. 3–4, 82–90.

7932:

DeClaris, N. Prof. Dr. Balthasar van der Pol: in memoriam. IRE Trans. **CT-7** (1960), 360–361.

LOGIC AND FOUNDATIONS

See also 8056, B9378.

7933:

Fisk, Milton. Contraries. Methodos **11** (1959), 319–351.

Nennt man "konträr" jedes Paar von Prädikaten A, B, für das von den Aussagen Ax, Bx notwendigerweise eine, möglicherweise beide falsch sind, so erhält man einen zu weiten Begriff. z.B. wären hiernach auch 'heiss' und 'lauwarm', 'rot' und 'grün' konträr. Verf. definiert vier Typen konträrer Prädikate: Symmetrische Glieder einer Reihe, z.B. heiss-kalt, konverse Relationen, z.B. grösser-kleiner. Reiz-Reaktion, z.B. fragen-antworten, zeitliche Umkehrungen, z.B. kommen-gehen. Durch Zusammensetzung entstehen weitere Typen.

P. Lorenzen (Kiel)

7934:

Zlot, William Leonard. The principle of choice in pre-axiomatic set theory. Scripta Math. **25** (1960), 105–123.

This is an interesting expository and historical article about the axiom of choice and its role in set theory. It sketches Zermelo's two proofs of the well-ordering theorem, and describes the controversies they aroused, as illustrated by the five letters on set theory written by Hadamard, Baire, Lebesgue and Borel and published in Bull. Soc. Math. France **33** (1905), 261–273. It contains numerous references.

O. Frink (Dublin)

7935:

Chauvin, André. Sur des modèles du calcul K_0 de Bochvar, avec ou sans égalité, et l'interprétation des paradoxes de la logique dans la théorie des ensembles élémentaires arithmétiques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 124–131.

The author introduces a system k_0 obtained from Bochvar's system K_0 [Mat. Sb. **15 (57)** (1944), 369–384; MR **7**, 46] by omitting the symbol for equality and the corresponding axioms. The calculi k_0 and K_0 have an interpretation in the extended propositional calculus. In consequence each sentence in k_0 and K_0 can be reduced to a Boolean expression. This procedure enables one to examine the consistency of various extensions of k_0 and K_0.

The above-mentioned calculi also have an interpretation in the theory of elementary arithmetic sets. The paradoxes, consisting of the fact that the adjunction to k_0 or K_0 of some definitions (such that the definiendum contains only one symbol which does not belong to this system, and the definiens contains only symbols from this system) renders the system inconsistent, can be interpreted as signifying that some elementary arithmetic sets are not recursively enumerable.

H. Rasiowa (Warsaw)

7936:

Grzegorczyk, Andrzej. ★Zagadnienia rozstrzygalności [Decision problems]. Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 142 pp. zł. 16.00.

The aim of this book is to present in a popular way the elements of the theory of recursive functions and the Gödel incompleteness theorem. Starting from the arithmetical definition for the class of recursive functions (the smallest class of functions containing zero and the successor function and closed with respect to the operations of superposition, primitive recursion and effective minimum), the author proves the representability of recursive functions in elementary arithmetic and hence the essential undecidability and the incompleteness of elementary arithmetic. The very clear and simple presentation makes the book easy reading even for non-mathematicians. At the same time much additional information on related problems and results, with references, is supplied for more advanced readers.

H. Rasiowa (Warsaw)

7937:

Shoenfield, J. R. An uncountable set of incomparable degrees. Proc. Amer. Math. Soc. **11** (1960), 61–62.

Verf. zeigt im Anschluss an Kleene und Post [Ann. of Math. (2) **59** (1954), 379–407; MR **15**, 772] durch eine induktive Definition, dass es zu jeder Folge $a_0, a_1, \cdots$ von Graden rekursiver Unlösbarkeit einen Grad b gibt, der mit allen a_n unvergleichbar ist. Hieraus folgt (mengentheoretisch) die Existenz überabzählbar-vieler Grade.

P. Lorenzen (Kiel)

7938:

Dekker, J. C. E.; Myhill, J. Recursive equivalence types. Univ. California Publ. Math. **3**, 67–213 (1960).

This monograph is an impressive part of the contemporary tendencies towards application of constructive methods (in definitions rather than in proofs) to mathematics. It can be regarded as a constructive theory of cardinality.

The completion of this monograph, announced in some earlier papers by the authors, has a four-year history; so, although its first version was ready in 1956, it appeared at the end of 1960. In the meantime some of its content was made known by Myhill [Bull. Amer. Math. Soc. **64** (1958), 373–376; MR **21** #7] and Dekker [Math. Z. **70** (1958), 113–124, 250–262; MR **20** #5133, 5134]. Nevertheless, the reviewer is inclined to include some repetitions, viz., for chapters I and II, whose content was partly covered in the review of the paper by Myhill [loc. cit.].

Notations: $m, n, p, q, r, \cdots$ are natural numbers; $\alpha, \beta, \gamma, \cdots$ sets (of natural numbers); $\alpha', \beta', \gamma', \cdots$ complements and $A, B, C, \cdots$ classes of such sets; $X, Y, Z, \cdots$ special classes of such sets, called isols; $\emptyset$ the empty set; ε the set of all (natural) numbers; E the class of all recursive sets; Q the class of all finite sets; and F the class of all recursive enumerable (r.e.) sets. $f, g, h, \cdots$ denote functions, and δf the domain, ρf the range of f.

The monograph consists of 15 chapters and an appendix.

Chapter 1 states some basic definitions and elementary propositions. α is equivalent to β ($\alpha \sim \beta$), if there exists a 1-1 partial function f such that $\alpha \subset \delta f$ and $f(\alpha)=\beta$; α is isomorphic to β ($\alpha \cong \beta$) if there exists a recursive function f, which maps ε 1-1 onto itself (a recursive permutation), such that $f(\alpha)=\beta$; α is recursively equivalent to β ($\alpha \simeq \beta$) if there is a partial recursive 1-1 function f (a partial isomorphism) such that $\alpha \subset \delta f$ and $f(\alpha)=\beta$. Req $\alpha=\{\beta|\beta\simeq\alpha\}$ is called a recursive equivalence type (R.E.T.). For finite sets $\alpha \cong \beta \leftrightarrow \alpha \simeq \beta \leftrightarrow \alpha \sim \beta$. For r.e. sets $\alpha \cong \beta \leftrightarrow (\alpha \simeq \beta \,\&\, \alpha' \simeq \beta')$. So two infinite r.e. sets are isomorphic if and only if their complements are recursively equivalent, and the classification of the sets in $F-E$ up to isomorphism can be reduced to the classification of the R.E.T.'s of their complements. This explains why special interest should be paid to R.E.T.'s of sets which are not r.e. It is astonishing how rich and unfamiliar is the collection Ω of all R.E.T.'s.

Chapters II and III introduce addition of R.E.T.'s. $A+B=\mathrm{Req}(\alpha+\beta)$ for $\alpha \in A$, $\beta \in B$ and α separable from β ($\alpha|\beta$). $A \leqq B \leftrightarrow (\exists C)(A+C=B)$. $0\cdot A=0$, $(n+1)\cdot A=n\cdot A+A$, where $0=\mathrm{Req}(\emptyset)$; $\infty\cdot\alpha=j(\varepsilon\times\alpha)$, where $j(x, y)=x+(x+y)(x+y+1)/2$; $\infty\cdot A=\mathrm{Req}(\infty\cdot\alpha)$ for $\alpha\in A$. The main results are the following: $A+B=A \leftrightarrow A \geqq \infty\cdot B$; $(\Omega, +)$ has the finite refinement property, i.e., $A+B=C+D \to (\exists E_1, E_2, E_3, E_4)(A=E_1+E_2 \;\&\; B=E_3+E_4 \;\&\; C=E_1+E_3 \;\&\; D=E_2+E_4)$. Using some results of Tarski's cardinal algebras, many elementary theorems about $A+B$, $A \cup B$ (the least upper bound), $A \cap B$ (the greatest lower bound) are derived.

The most important is the definition of isols. $A \in \Omega$ is quasi-finite if $A+B=A+C \to B=C$; an R.E.T. is an isol if it is quasi-finite. The collection of all isols is denoted by

Λ. Defining $(\Delta, +)$ to be an ideal in $(\Omega, +)$ if $\Delta \subset \Omega$, Δ is closed under addition and $A \in \Omega$ & $A \leqq B$ & $B \in \Delta \rightarrow A \in \Delta$, it follows that $(\Lambda, +)$ is an ideal in $(\Omega, +)$.

Chapter IV describes the nature of the ideal $(\Lambda, +)$. With $L_0 = \mathrm{Req}(\emptyset)$, $L_n = \mathrm{Req}(1, 2, \cdots, n)$ and $\Lambda_Q = \{L_n | n \in \varepsilon\}$, $(\Lambda_Q, +)$ is shown to be isomorphic with $(\varepsilon, +)$ (under the mapping $L_n \leftrightarrow n$). Thus, $(\varepsilon, +)$ is a subsystem of $(\Lambda, +)$, and L_n is identified with n. Most of this chapter (and other chapters following) is devoted to a search for properties of natural numbers which can be extended to isols.

A set is isolated if it is not recursively equivalent to a proper subset. Isols are R.E.T.'s of exactly those sets which are isolated. They can be characterized by each of three properties: $X < X + A$ for $A \neq 0$; $X < X + 1$; $(\forall B)(\forall C)(X + B = X + C \rightarrow B = C)$. The cardinalities of ε (i.e., Λ_Q), Λ, Ω are $\aleph_0$, c, c, respectively.

Chapter V states the definition of $A - n$. It is shown first that for every $A \neq 0$ there exists a unique B for which $B + 1 = A$. Then $A - n$ is defined by: $A - 0 = A$, $A - (n+1) = (A - n) - 1$ for every $n + 1 \leqq A$. If A is not an isol, $A - n = A$. But for an isol X, we have $\cdots < X - 2 < X - 1 < X$. Thus, every isol initiates a series of the type $\omega^* + \omega$, $\cdots < X - 2 < X - 1 < X < X + 1 < X + 2 < \cdots$, of different R.E.T.'s. It is shown also that: $(n+1) \cdot X = (n+1) \cdot Y \rightarrow X = Y$; $(n+1) \cdot X + Z = (n+1) \cdot Y \rightarrow (\exists U)\,(Z = (n+1) \cdot U)$.

Chapter VI gives some existence theorems. The R.E.T. A is indecomposable if $B + C = A \rightarrow B \in \varepsilon \vee C \in \varepsilon$, otherwise decomposable. If $\square$ is a non-empty at most denumerable collection of infinite R.E.T.'s, there exist c isols which are indecomposable, mutually incomparable and incomparable with each member of $\square$. Also, if $\square$ is a denumerable collection of R.E.T.'s [isols], there exists an R.E.T. [isol] which exceeds all members of $\square$.

Chapter VII is concerned with some ideals in Ω. The ideal $\square$ is called principal if it contains an element A such that $\square$ = the collection of all finite sums of members of the set $\{B | B \leqq A\}$. With $\Omega_F = \{\mathrm{Req}\ \alpha' | \alpha \in F\}$, Z = the class of all simple sets, $\Lambda_Z = \varepsilon + \{\mathrm{Req}\ \alpha' | \alpha \in Z\}$, and Λ_Σ = collection of all isols which can be expressed as finite sums of indecomposable isols, it is shown that ε, Λ, Ω_F, Λ_Z, Λ_Σ are ideals in Ω and that Ω_F is a principal ideal (and Λ and Λ_Σ are not).

The principal ideal Ω_F is significant for classification of unsolvable decision problems, and in chapter VIII it is studied in detail. Req α' is called the type of the r.e. set α. A classification of types is given; it is proved that the R.E.T. of every productive set is $\geqq H_1 = \mathrm{Req}\ k'$, where k is Post's (or any other) creative set.

The multiplication of R.E.T.'s is studied in chapter IX: $A \cdot B = \mathrm{Req}\,(j(\alpha \times \beta))$, $\alpha \in A$, $\beta \in B$. A^n and A^∞ are introduced in chapter X: $A^0 = 1$, $A^{n+1} = A^n \cdot A$; A^∞ is defined so as to play in multiplication a role similar to that of $\infty \cdot A$ in addition. For instance, $A \cdot B = A \leftrightarrow A \cdot B^\infty = A$. Chapter XI considers subtraction, which is unique only for isols, and the extension of $[\Lambda, +, \cdot]$ to the ring $[\Lambda^*, +, \cdot]$ of isolic integers.

In chapter XII, the following fundamental (and deep) theorem is proved: $X \cdot Z = Y \cdot Z$ & $Z \neq 0 \rightarrow X = Y$ (only for isols!). Thus, it is possible to define $X/Y = U \leftrightarrow Y \cdot U = X$.

Chapter XIII contains the theory of prime R.E.T.'s. A is prime if $A \geqq 2$ and $B \cdot C = A \rightarrow B = 1 \vee C = 1$. For every at most denumerable collection $\square$ of R.E.T.'s there exists a prime exceeding every member of $\square$. If X is an infinite prime isol, then X^2 has no finite divisors greater than 1.

Exponentiation A^B is treated by means of combinatorial functions [see Myhill, op. cit.]. It is shown that Ω_F, Λ, Λ_Z are closed under exponentiation.

Some theorems about the kind of constructivity used by the authors and the proof of $\alpha \cong \beta \leftrightarrow (\alpha \simeq \beta$ & $\alpha' \simeq \beta')$ for any sets α, β (not only r.e.) are given in chapter XV and in the appendix, respectively.

This review is a very brief exposition of the rich content of this significant monograph. The authors have introduced many new ideas in the proofs, combining freely the algebraic and set-theoretic methods with those of the theory of recursive functions. Many open (and not trivial) problems increase the interest of the theory.

Some theorems and proofs of S. Tennenbaum, Dana Scott, R. Friedberg and A. Nerode appear here for the first time. The authors state that the proof of the theorem in the Appendix was also obtained independently by Carol Karp.

Errata: P. 73, l. 14 from bottom, read $\alpha_2 + \beta_2 \simeq \alpha_1 + \beta_1$; p. 97, l. 21 from bottom, read $\delta = g(\gamma) \subset \beta_1$; p. 137, l. 12 from bottom, read A^∞. There are other minor misprints.

V. Vučković (Belgrade)

7939:

Vučković, Vladeta. Partially ordered recursive arithmetics. Math. Scand. 7 (1959), 305–320.

An arithmetic with a simple zero 0 and a finite or denumerably infinite number of successor operations $S_0, S_1, \cdots$ is treated by the methods of Goodstein [*Recursive number theory*, North-Holland, Amsterdam, 1957; MR **21** #1272]. The notion of a primitive recursive function is extended to this case in an obvious way. It is postulated that $S_\nu S_\mu x = S_\mu S_\nu x$; therefore, after each definition of a function F, it must be verified that $F(S_\nu S_\mu x) = F(S_\mu S_\nu x)$. $S_0 0$ is written as 1. Functions $x + y$, $x \cdot y$, predecessor functions $P_\nu x$, the difference $x \dot{-} y$ and the symmetric difference $|x, y| = (x \dot{-} y) + (y \dot{-} x)$ are defined. The scheme of double recursion is slightly different from the usual one. Logical connectives between equations can be defined in different ways, leading to different forms of logic. (1) Let $\alpha(x)$ be defined by $\alpha(0) = 0$, $\alpha(S_\nu x) = 1$. Let $a = b$ be called true if $\alpha(|a, b|) = 0$ is provable, and false if $\alpha(|a, b|) = 1$. Then Goodstein's treatment of logic can be taken over. (2) Let $a = b$ be called true if $|a, b| = 0$ and false if $1 \dot{-} |a, b| = 0$. Now the same method leads to a modification of intuitionistic logic, where $(p \supset q) \wedge (p \supset \sim q) \supset \sim p$ does not hold, but $\sim p \vee \sim\sim p$ does hold.

A. Heyting (Amsterdam)

7940:

Péter, R. Über die Partiell-Rekursivität der durch Graphschemata definierten zahlentheoretischen Funktionen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 41–48.

This paper contains a more complete proof than has appeared earlier of the following theorem: The set of partial recursive functions coincides with the set of those number-theoretic functions definable by a Normalschema; moreover, the general recursive functions coincide with those whose values for all natural number k-tuples are given by a Normalschema. A Normalschema is defined as a finite connected graph. Two kinds of vertices are distinguished: the mathematical vertex having one exit edge, and the logical vertex having two. The present proof is more complete in the sense that functions corresponding

to logical vertices are provided that are similar to functions provided in an earlier proof only for mathematical vertices.
E. J. Cogan (Bronxville, N.Y.)

7941:
Liu, Shih-Chao. Proof of a conjecture of Routledge. Proc. Amer. Math. Soc. **11** (1960), 967–969.

The author proves a conjecture of N. A. Routledge [Proc. Cambridge Philos. Soc. **49** (1953), 175–182; MR **14**, 714] that not every general recursive function is expressible in the form $\phi(g(a))$ with ϕ primitive recursive and $g(a)$ defined by the ordinal recursion

$$g(n) = m, \quad g(a) = h(a, g(\delta(a))), \qquad a \neq n,$$

where $h(a, g)$, $\delta(a)$ are primitive recursive and $\delta(a) < a$, for $a \neq n$, in a well-ordering, of type ω, of the natural numbers with n as first element. It is shown that for any general recursive $\psi(a)$, primitive recursive $\phi(a)$ and ordinal recursive $g(a)$, an argument a can be constructively determined for which $\psi(a) \neq \phi(g(a))$, provided that the well-ordering of type ω involved in the definition of $g(a)$ is itself constructively given.
R. L. Goodstein (Leicester)

7942:
Meigne, Maurice. ★La consistance des théories formelles et le fondement des mathématiques. Librairie Scientifique et Technique Albert Blanchard, Paris, 1959. 115 pp. 12.00 NF.

SET THEORY

See also B9284.

7943:
Rey Pastor, Julio. ★Apuntes de teoria de los conjuntos abstractos [Notes on abstract set theory]. Instituto de Matematicas, San Luis (Argentina), 1957. ii + 55 pp.

Elements of set theory.

7944:
Isbell, J. R. A definition of ordinal numbers. Amer. Math. Monthly **67** (1960), 51–52.

The author calls a set Y transitive if every element of Y is a subset of Y, and he defines a set X to be an ordinal number if every transitive proper subset of X is an element of X. {This definition is closely related to one of Bernays [H. Bachmann, *Transfinite Zahlen*, Springer, Berlin, 1955; MR **17**, 134; p. 20].}
F. Bagemihl (Ann Arbor, Mich.)

COMBINATORIAL ANALYSIS

7945:
Stein, Sherman K. The mathematician as an explorer. Sci. Amer. **204** (1961), no. 5, 148–158.

A non-technical account of the author's experiences with the problem of constructing a sequence of zeros and ones in which every ordered n-tuple of zeros and ones occurs (consecutively) exactly once; see I. J. Good, J. London Math. Soc. **21** (1946), 167–169; N. G. de Bruijn, Nederl. Akad. Wetensch. Proc. **49** (1946), 758–764 [MR **8**, 430, 247].

7946:
Robinson, W. J. The Josephus problem. Math. Gaz. **44** (1960), 47–52.

The Josephus problem [W. W. Rouse Ball, *Mathematical recreations and essays*, 11th ed., Macmillan, New York, 1947; MR **8**, 440; pp. 32–36] is essentially this: If 41 objects $a_1, a_2, \cdots, a_{41}$ are placed in a circle and every third object is removed (counting around in the same sense all the time), then a_{31} is the last object to be removed. In this note the author describes and solves several related problems and generalizations.
W. Moser (Winnipeg, Man.)

7947:
Joseph, A. W.; Bizley, M. T. L. The two-pack matching problem. J. Roy. Statist. Soc. Ser. B **22** (1960), 114–130.

Given a pack of cards consisting of a_i identical cards of the ith kind ($i = 1, 2, \cdots, n$), the number of derangements is obtained by applying an adaptation of the Laguerre polynomial considered as an operator. This leads to explicit formulas and generating functions for the number of "r-hits", that is, the number of distinct permutations which produce exactly r coincidences upon being matched with a fixed arrangement of a second pack of identical composition. Certain properties of the adapted Laguerre polynomials are deduced, from which recurrence formulas are developed for the number of derangements, both generally and also when $a_i = a$ ($i = 1, 2, \cdots, n$). It is shown how to build up a hit-generating function stage by stage by a simple numerical process. Computation methods previously given [e.g., Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958; MR **20** #3077] involve the calculation of many coefficients with alternating signs, most of which are much larger than the final result. The process developed here is purely additive and thus avoids the need to calculate any number larger than the final results. Tables are given of the number of derangements for $n = 3, 4, 5$ and $1 \leq a_i \leq 5$ ($i = 1, 2, \cdots, n$), and for $a = 2, 3, 4, 5$, where n ranges from 2 up to 12, 9, 7 and 6, respectively, and of the number of r-hits for $a = n = 3, 4, 5$ and $r = 0, 1, \cdots, n^2$.
T. N. E. Greville (Kensington, Md.)

7948:
Eperson, D. B. Magic square patterns. Math. Gaz. **43** (1959), 273–274.

7949:
Hoffman, A. J. On the uniqueness of the triangular association scheme. Ann. Math. Statist. **31** (1960), 492–497.

A partially balanced incomplete block design with two associate classes is said to be triangular if the number of treatments, v, is $n(n-1)/2$ for some integer n, and the association scheme is obtainable as follows. Let the v treatments be regarded as all possible arcs of the graph determined by n points; let the first associates of any arc (= treatment) be all arcs each of which shares exactly one end point with the given arc; let the second associates

of any arc be all arcs each of which does not share an end point with the given arc and does not coincide with the given arc. Then (a) the number of first associates for any treatment is $2(n-2)$; (b) if θ_1 and θ_2 are two treatments which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is $n-2$; (c) if θ_1 and θ_2 are second associates, the number of treatments which are first associates of both θ_1 and θ_2 is 4. It is natural to inquire if conditions (a), (b), (c) imply that the $v=n(n-1)/2$ treatments can be represented as arcs on the graph determined by n points in the manner described.

Connor [same Ann. **29** (1958), 262–266; MR **20** #3620] has shown this is the case for $n \geqq 9$. The present paper shows that the result does not hold for $n=8$, but does hold for $n=7$. Since Shrikhande has shown that the result holds for $n \leqq 6$ [ibid. **30** (1959), 39–47; MR **21** #1673], the question is now settled. The author notes that the results of this paper have also been obtained, using different methods, by Chang [#7950]. Chang has also shown that there are exactly three counterexamples when $n=8$, namely, for $v=28$, $n_1=12$, $n_2=15$, $p_{11}{}^2=4$ [#7951].

R. G. Stanton (Waterloo, Ont.)

7950:

Chang, Li-chien. The uniqueness and nonuniqueness of the triangular association schemes. Sci. Record (N.S.) **3** (1959), 604–613.

Consider an association scheme for a partially balanced incomplete block design with two associate classes having the parameters

$$(*) \qquad n_1 = 2n-4, \quad p_{11}{}^1 = n-2, \quad p_{11}{}^2 = 4.$$

The author shows that these parameters necessarily imply the existence of a unique triangular association scheme for $n \neq 8$. When $n=8$ the author presents an association scheme with the above parameters which is not a triangular association scheme. This same result was recently proved independently by A. J. Hoffman, the case for $n \geqq 9$ by Connor and for $n=5, 6$ by Shrikhande [review #7949 and references therein].

M. Zelen (College Park, Md.)

7951:

Chang, Li-chien. Association schemes of partially balanced designs with parameters $v=28$, $n_1=12$, $n_2=15$ and $p_{11}{}^2=4$. Sci. Record (N.S.) **4** (1960), 12–18.

In the paper reviewed above the author has proved that an association scheme with the parameters (*) does not necessarily have a triangular association scheme for $n=8$. This paper presents the possible four distinct association schemes for $n=8$. Only one of these schemes is triangular.

M. Zelen (College Park, Md.)

7952:

Kesava Menon, P. Difference sets in Abelian groups. Proc. Amer. Math. Soc. **11** (1960), 368–376.

Let A be a finite Abelian group of order v and let D be a subset of k different elements of A, and suppose that an integer λ exists such that $k(k-1)=\lambda(v-1)$. Further suppose that in the group ring A^* of A over the integers we have $\Delta=\sum_{x\in D} x$, $\Delta'=\sum_{x\in D} x^{-1}$, and $\Delta\Delta'=(k-\lambda)1+\lambda\sum_{x\in A} x$. Then D is a difference set in the group A, and leads to a v, k, λ symmetric block design which has A as a natural group of automorphisms. A multiplier of D, following Bruck [Trans. Amer. Math. Soc. **78** (1955), 464–481; MR **16**, 1081] is an automorphism φ of A such that for some $x \in A$, $\varphi(D)=xD$. A particular kind of automorphism is φ_t, given by $\varphi_t(x)=x^t$, where t is an integer such that $(t,v)=1$. The following theorem is proved: If n_1 is a divisor of $n=k-\lambda$ such that $(n_1, v)=1$ and $n_1>\lambda$, and if there exists an integer t such that for each prime p dividing n there is a number j such that $p^j \equiv t \pmod v$, then the automorphism φ_t is a multiplier of the difference set D. This generalizes a similar theorem of the reviewer [Proc. Amer. Math. Soc. **7** (1956), 975–986; MR **18**, 560] proved when A is cyclic. The proof makes use of the characters of A.

An example of a different kind of multiplier is given where $v=2^{2n}$, $k=2^{2n-1}-2^{n-1}$, $\lambda=2^{2n-2}-2^{n-1}$ and A is the elementary Abelian group of order 2^{2n}. When A is taken as the $2n$-dimensional vector space over the field with two elements, D is the set of elements with a number of ones $\equiv 0, 1 \pmod 4$ or its complement, and the multipliers are the permutations of the components of the vectors.

Marshall Hall, Jr. (Pasadena, Calif.)

7953:

Tinsley, M. F. Permanents of cyclic matrices. Pacific J. Math. **10** (1960), 1067–1082.

Let $A=[a_{ij}]$ be an $n\times n$ matrix with non-negative real entries. The permanent of A is defined by

$$P(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n},$$

when the summation extends over the $n!$ permutations of the integers $i_1, i_2, \cdots, i_n$. This paper investigates a certain class of matrices in which the permanent and determinant are equal in absolute value. This property is designated by $P(A)=|D(A)|$. Now let A be a (0, 1)-matrix with row and column sums equal to s. Let $\Sigma=[\sigma_{ij}]$ be a permutation submatrix of A. With Σ there is associated the permutation Σ' on 1, 2, $\cdots$, n given by $\Sigma'(i)=j$ if and only if $\sigma_{ij}=1$. Now let A be written as a sum of s permutation matrices $A=\pi_1+\pi_2+\cdots+\pi_s$. If $\pi_k'\pi_j'=\pi_j'\pi_k'$ for each j and k, then A is called abelian. If for $i=1, 2, \cdots, s$, $\pi_i'=(1, 2, \cdots, n)^{d_i}$, where $0 \leqq d_i < n$, then A is cyclic and is defined by the differences $d_1, d_2, \cdots, d_s$ mod n. Let C be the 7×7 cyclic matrix defined by the differences 0, 1, 3 mod 7. The central theorem of this paper is the following. Let A be an $n\times n$ abelian matrix with $s \geqq 3$ ones in each row and column. Then $P(A)=|D(A)|$ if and only if $s=3$, $n=7e$ and upon permutations of rows and columns A is transformed into the direct sum of C taken e times. The proof of this theorem is intricate and we cannot go into the details here. But this theorem, along with various corollaries, does give considerable insight into the behavior of the permanent in combinatorial investigations.

The concluding theorem in this paper asserts the following. Let A by a cyclic (0, 1)-matrix of order n defined by the differences 0, d_1, d_2 mod n. Then $P(A)=|D(A)|$ if and only if $n=7e$, $d_1=ed_1'$, $d_2=ed_2'$, where 0, d_1', d_2' is a perfect difference set mod 7. It has been conjectured that this theorem remains valid with the deletion of the word "cyclic". But in a communication to the reviewer, the author remarks that he has subsequently shown that the cyclic requirement is an essential part of the hypothesis.

H. J. Ryser (Columbus, Ohio)

7954:

Haber, Robert Morton. Term rank of 0, 1 matrices. Rend. Sem. Mat. Univ. Padova **30** (1960), 24–51.

Let $\mathfrak{A}(R, S)$ be the class of all n by m matrices whose entries consist of 0's and 1's; R is the vector of row-sums and S the vector of column sums. The term ranks of all the matrices of the class $\mathfrak{A}(R, S)$ lie between a minimum value $\tilde{\rho}$ and a maximum value $\bar{\rho}$. Ryser, Gale, and Higgins have previously published proofs of a necessary and sufficient condition on R and S that the set $\mathfrak{A}(R, S)$ be non-vacuous. Ryser has also shown that it is possible to get from one matrix of the set to another by successive replacements of $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and vice versa. Ryser has also given a simple formula for $\bar{\rho}$. In this paper the author re-proves these results. {The reviewer can see no obvious advantage or novelty of approach in his proofs.} He also obtains an algorithm for computing $\tilde{\rho}$, although it is extremely complicated. He concludes with the construction of a few numerical examples.

N. S. Mendelsohn (Winnipeg, Man.)

7955:

Carlitz, L. Some formulas of Jensen and Gould. Duke Math. J. **27** (1960), 319–321.

The author establishes the following two theorems. (I) If $\{Q_k(x)\}$ is a sequence of polynomials, $Q_0(x)=1$, $\deg Q_k(x)=k$, such that

$$\sum_{k=0}^{n} Q_k(\alpha+\beta k)Q_{n-k}(\gamma-\beta k) = \sum_{k=0}^{n} \beta^k Q_{n-k}(\alpha+\gamma-k)$$

for all α, β, γ, then $Q_n(\alpha)=\binom{\alpha}{n}$ $(n=0, 1, 2, \cdots)$. (II) If $\{Q_k(x)\}$ is a sequence of polynomials, $Q_0(x)=1$, $\deg Q_k(x)=k$, such that

$$\sum_{k=0}^{n} Q_k(\alpha+\beta k)Q_{n-k}(\gamma-\beta k) = \sum_{k=0}^{n} \beta^k Q_{n-k}(\alpha+\gamma)$$

for all α, β, γ, then $Q_n(\alpha)=\alpha^n/n!$ $(n=0, 1, 2, \cdots)$. Theorem I was suggested to the author by a related expansion due to Jensen [Acta Math. **26** (1902), 307–318]. Theorem II was suggested by a formula of H. W. Gould [Duke Math. J. **27** (1960), 71–76; MR **22** #5585]. The proofs depend upon mathematical induction.

A. L. Whiteman (Princeton, N.J.)

7956:

Tutte, W. T. An algorithm for determining whether a given binary matroid is graphic. Proc. Amer. Math. Soc. **11** (1960), 905–917.

The algorithm develops naturally from the author's characterizations [Trans. Amer. Math. Soc. **88** (1958), 144–174; **90** (1959), 527–552; MR **21** #336, 337]. Determining whether a given matroid is binary is not difficult. Further, by duality, this algorithm applies also to determining whether a given graph is planar. However, though the method presented here is a reasonable algorithm for determining whether the matroid is graphic, the indications given for going on to construct the graph do not seem to describe a reasonable algorithm.

J. R. Isbell (Seattle, Wash.)

7957:

Ore, Oystein. Sex in graphs. Proc. Amer. Math. Soc. **11** (1960), 533–539.

If (b, c) is an edge in a directed graph G, call b the "parent" of c. If the vertices of the graph G can be divided into two non-overlapping classes M, F ("sexes"), such that no vertex has two or more parents of the same sex, the graph has a "sex-dichotomy". The necessary and sufficient condition for the existence of a sex-dichotomy is that every sequence of edges of the form (a_0, a_1), (a_2, a_1), $\cdots$, (a_{2r}, a_{2r+1}), (a_{2r+2}, a_{2r+1}), $\cdots$, (a_{2n-2}, a_{2n-1}), (a_{2n}, a_{2n-1}) with $a_{2n}=a_0$ has n even. The numbers of ways of assigning sexes to the vertices is then 2^k, where k is the number of connected components of a graph $G \cdot G^*$ easily obtained from G.

C. A. B. Smith (London)

7958:

Rosenblatt, David. On some aspects of models of complex behavioral systems. Information and decision processes, pp. 62–86. McGraw-Hill, New York, 1960.

This is a concise and compact review of some of the principal results in the theory of linear graphs and the isomorphic algebra of non-negative square matrices under Boolean rules of addition and multiplication. The author goes on to show the application of these results to the representation of "complex behavioral systems", e.g., large-scale interindustrial "input-output" models, interaction models representable by Markov chains, and communication models. There are several interesting allusions to some early precursors of the modern formulations, for example, to those of the eighteenth-century French thinkers Quesnay (biologist-philosopher) and Isnard (economist), and even to a fifteenth-century Venetian mathematician Fra Luca Pacioli (first published formalization of double-entry bookkeeping?). The short but well-selected bibliography can serve as a guide to those interested in the mathematics of system structure.

A. Rapoport (Ann Arbor, Mich.)

ORDER, LATTICES

7959:

Popruženko, J. Sur la vitesse de la croissance des suites infinies d'entiers positifs. II. Espace des vitesses. Fund. Math. **48** (1959/60), 71–78.

The author first considers an abstract irreflexive partial ordering relation ρ on a space $\mathcal{N}$ and defines, for an increasing transfinite sequence a_ξ, $\lim a_\xi = b$ if b is a least strict upper bound of $\{a_\xi\}$, and defines the limit to be a strong limit if $c\rho b$ implies $c\rho a_\xi$ for some ξ. He shows that, if every finite or denumerable sequence does not have a limit, then every element which is not minimal is a limit. Furthermore, if no finite or denumerable set has a least strict upper bound, and if $\mathcal{N}$ has cardinality $\aleph_\mu$, then if every transfinite sequence converging to b has cardinality $\aleph_\mu$ at least one sequence converges strongly to b, and the converse holds if $\aleph_\mu$ is regular. These results follow from results in a previous paper by the author [Fund. Math. **46** (1959), 235–242; MR **21** #11]. Introducing an equivalence relation in the obvious manner, he proves that the results on limits go over unchanged.

The remainder of the paper is devoted to sequences of positive integers introduced in the paper cited. For the space of equivalence classes, he defines derived set by means of the limit relation and shows that the derived set of a countable union of sets is the union of the derived sets. Assuming the continuum hypothesis, he shows that this yields a dense-in-itself T_1 space in which there exist sets with only one additional point in their closures.

H. Rubin (E. Lansing, Mich.)

7960:

Ricabarra, Rodolfo A. ★Conjuntos ordenados y ramificados (Contribución al estudio del problema de Suslin) [Ordered and ramified sets (Contribution to the study of Souslin's problem)]. Instituto de Matematica, Universidad Nacional del Sur, Bahía Blanca, 1958. 357 pp. 400 pesos; \$6.00.

This book presents a welter of information on the so-called (K)-sets (see below) and their relation to ramified tables and Souslin's problem, many of the ideas of which are due to Kurepa. As such it fills a definite gap in, and is a welcome addition to, the mathematical literature. Because of the specialization of the material, the book is recommended only for the specialist. It definitely is not for the student who wishes to learn general set theory.

There are nine main chapters plus a tenth which consists of bibliographic notes and comments. There is no index (a serious deficiency in view of the multitude of terms) and a much too short bibliography. The author's style is pleasant and leisurely. One notable exception: On pages 60–61 there is a sentence eleven lines long which this reviewer never did understand. A brief résumé, chapter by chapter, will now be given.

Chapter 1 reviews simply and partially ordered sets, ramified tables, and ramified sequences. (By definition, a ramified sequence C is a ramified table with the following condition: Given any element a in C and any ordinal number α less than the length of the table, there exists an element b in the α-range comparable with a.)

Chapter 2 introduces simply ordered sets of type (K). A simply ordered set is said to be a (K)-set (or of type (K)) if it satisfies the following eight conditions. (1) It has a first element. (2) It has a last element. (3) It is dense. (4) Every denumerable subset is "ralo" in the order topology. (5) It satisfies the first countability axiom. (6) It is σ-complete. (7) Any family of disjoint sections, not points, which are linearly ordered, is at most denumerable. (8) It has a topologically dense subset of power $\leqq\aleph_1$. It is shown that Souslin's problem consists in deciding if there exists a (K)-set with a special property. Finally, the Denjoy-Kurepa type, denoted by κ_1 is introduced and shown to be a (K)-set.

Chapter 3 deals with triadic developments of (K)-sets. By definition, a triadic development of a (K)-set K is a family $\{T_\alpha|1\leqq\alpha<\omega_1\}$ of subsets of K with the following properties. (1) T_α is a closed subset of K, of triadic type as a subtype of K, whose extremes coincide with that of K. (2) If $E_\alpha{}^i$, $E_\alpha{}^d$ are the sets formed by the lower, upper (respectively) extremes of the jumps of T_α, then for $0<\beta<\alpha<\omega_1$ both $T_\beta\subseteq T_\alpha$ and $T_\beta\cap E_\alpha{}^i=T_\beta\cap E_\alpha{}^d=\emptyset$. (3) If α is a limit number then $T_\alpha=(\bigcup_{\beta<\alpha}T_\beta)^-$. (4) $K=\bigcup_{\alpha<\omega_1}T_\alpha$. Among other results, triadic developments are characterized as increasing families of triads with certain properties. The concepts of lower rational and upper rational, range of points, and ramified table of triadic intervals (with respect to a fixed triadic development of a (K)-set) are introduced and some basic properties developed.

Chapter 4 is a miscellaneous collection of facts about subsets of (K)-sets. Some of the subsets studied are discrete, rarified, topologically dense, of type (K), complete, etc.

Chapter 5 concerns homogeneous (K)-sets. A flood of definitions and results are given here. A sample is this: after defining a (K)-semigroup it is shown that the theory of homogeneous (K)-sets reduces to the theory of (K)-semigroups.

Chapter 6 relates to bounded types. In particular, bounded (K)-sequences, ramified tables of natural numbers, and homogeneous bounded types come under discussion.

Chapter 7 is concerned with the classification of (K)-sets by the number and density of their gaps. It is a fact that a (K)-set has at most $\aleph_2$ gaps (although none with exactly $\aleph_2$ are known). Four major classes are discussed, to wit, (K_0), (K_r), (K_1), and (K_2). (K_0) is the class of (K)-sets which are (ordinally) complete; (K_r) is the class whose set of gaps is a non-empty rarefied set; (K_1) is the class of (K)-sets such that any interval not a point has $\aleph_1$ gaps; and (K_2) is the class such that any interval not a point has $\aleph_2$ gaps. A number of results about these classes, such as their cardinality, are then given.

Chapter 8 discusses the theory of the Denjoy-Kurepa type. The two main results here are (1) a characterization in terms of (K)-sets, and (2) any two triadic developments of κ_1 are isomorphic.

Chapter 9 concerns various equivalences and implications of the Souslin problem.

S. Ginsburg (Santa Monica, Calif.)

7961:

Ohkuma, Tadashi. Comparability between ramified sets. Proc. Japan Acad. **36** (1960), 383–388.

A non-empty ramified set X is called perfectly irresoluble if for each element a in X the set $\{x|x>a, x \text{ in } X\}$ is not totally ordered. A set is called resoluble if it contains no non-empty subset which is perfectly irresoluble. For each regular ordinal number $\beta>0$, let R_β denote the family of ramified sets of power $<\aleph_\beta$, and S_β the family of resoluble sets in R_β. For ramified sets X and Y write $X\leqq Y$ if there is a mapping f of X into Y such that $x_1<x_2$ implies $f(x_1)<f(x_2)$ for all x_1 and x_2, and $X\equiv Y$ if $X\leqq Y$ and $Y\leqq X$. The following three theorems are stated and outlines of the proofs given. (A) (i) S_β is well-ordered by $\leqq$ under identification of equivalent sets. (ii) If either X or Y is in S_β, then $X\leqq Y$ or $Y\leqq X$. (B) The continuum hypothesis implies (i) there exist ramified sets for which neither $X\leqq Y$ nor $Y\leqq X$ hold, and (ii) there exists a sequence $\{X_n|n=1, 2, \cdots\}$ of non-equivalent ramified sets for which $X_n<X_{n+1}$. (C) R_1 is well-ordered by $\leqq$ under identification of equivalent sets.

S. Ginsburg (Santa Monica, Calif.)

7962:

Benado, Michaïl. Sur une caractérisation abstraite des algèbres de Boole. I. C. R. Acad. Sci. Paris **251** (1960), 622–623.

This note consists almost entirely of abstract definitions. A partially ordered set P is said to have a geometric

structure with respect to a division relation γ and multiplication relation σ [as defined by the author, same C. R. **247** (1958), 2265–2268; MR **20** #5745] if there exist a, b, d, m in P such that $d\gamma\{a, b\}$ and $m\sigma\{a, b\}$. Twelve types of such structures are defined; for instance: analytic; closed; complemented; modular in various senses.

P. M. Whitman (Silver Spring, Md.)

7963:

Benado, Michail. Sur une caractérisation abstraite des algèbres de Boole. II. C. R. Acad. Sci. Paris **251** (1960), 835–836.

This note states relationships between the various types of partially ordered sets P with geometric structure, defined in #7962. These generalize the author's previous work on multilattices [e.g., Czechoslovak Math. J. **5 (80)** (1955), 308–344; MR **17**, 937]. In particular, certain combinations suffice to make P a Boolean algebra. Details are to be published elsewhere.

P. M. Whitman (Silver Spring, Md.)

THEORY OF NUMBERS

See also 8006, 8245, 8248.

7964:

Browkin, J. On the periodicity of certain sequences of natural numbers. Wiadom. Mat. (2) **2**, 273–276 (1959). (Polish)

Let $\bar{n}$ be the natural number which is obtained by reversing the order of digits of a natural number n, written in the denary system of notation, and $f_s(n)=\overline{n+s}$ where s is a natural number. By U_s^{10} one denotes the set of all the sequences of the type (s fixed, denary system of notation):

$$(1)\qquad n, f_s(n), f_s^2(n), \cdots, f_s^r(n), \cdots,$$

where $f_s^r(n)=f_s[f_s^{r-1}(n)]$, $r\geqq 1$, $f_s^0(n)=n$. Further, let

$$A_{10}=\sum_{(s,10)=1} U_s^{10},\quad B_{10}=\sum_{(s,10)=10} U_s^{10},\quad C_{10}=\sum_{1\neq(s,10)\neq 10} U_s^{10}.$$

Every sequence (1) is an element of one of the sets A_{10}, B_{10}, C_{10}. Theorem 1: Every element of the set A_{10} is periodic. Theorem 2: If $U_s^{10}\subset B_{10}$, there exists a sequence in U_s^{10} which is not periodic. Theorem 3: Every element of U_s^p is periodic if and only if $(s, p)=1$, p a prime number.

J. W. Andrushkiw (Newark, N.J.)

7965:

Schinzel, A. On the periodicity of certain sequences of natural numbers. Wiadom. Mat. (2) **2**, 269–272 (1959). (Polish)

The author answers positively the question formulated by W. Sierpiński [Wiadom. Mat. (2) **2** (1959), 256–268; MR **22** #6755], whether for $s=3$, 7, 9, 11 and for any n the sequence (1), defined in Browkin's paper [see review #7964], is periodic. Theorem: For $s=3$, 7, 9, 11 and for any n there occurs in the sequence (1) a number less than 100. It follows that the set of all periods of all sequences considered in the theorem is finite.

J. W. Andrushkiw (Newark, N.J.)

7966:

Ferrier, A. Nombres idoines. Mathesis **69** (1960), 34–36.

A positive integer N is called idoneal if every odd number P which is representable uniquely in the form $P=a^2+Nb^2$ is necessarily a prime. These numbers were studied by Euler, who discovered a necessary and sufficient condition that a number be idoneal. He also found all idoneal numbers less than 10,000; there are 65 of them. This search was carried to 100,000 by Cullen and Cunningham who found no others. The present paper reports on a similar search up to 200,000. No additional idoneal numbers were found and it is natural to conjecture that there are no others. The author uses Euler's criterion and is able to eliminate many values of N by a systematic consideration of the last digit of N. He reports that it took him about 15 hours to complete the search.

H. W. Brinkmann (Swarthmore, Pa.)

7967a:

Pellegrino, Franco. Sulle funzioni analitiche numerico-integrali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 32–36.

7967b:

Pellegrino, Franco. Un teorema sulle funzioni analitiche numerico-integrali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 144–146.

Let f be any function on $N=\{1, 2, \cdots\}$ to the complex field and let $\psi=\psi(z)$ be a univalent analytic function defined on some region G containing $z_1=f(1)$. Let the sum and the product of two functions f_1, f_2 on N be defined by $(f_1+f_2)(n)=f_1(n)+f_2(n)$, $(f_1\times f_2)(n)=\sum_{d|n} f_1(d)f_2(n/d)$ $(n\in N)$.

The author discusses some known definitions of a function on N to be denoted by $\psi(f)$, and he gives a new definition of it in terms of an arbitrary function element of $\psi(z)$. He shows that his definition does not depend on the choice of this element.

In the second paper he proves that $\psi_1(f)\times\psi_2(f)=(\psi_1\cdot\psi_2)(f)$.

C. G. Lekkerkerker (Amsterdam)

7968:

Carlitz, L. Congruence properties of certain polynomial sequences. Acta Arith. **6** (1960), 149–158.

An announcement of the results of this paper appeared under the title "Arithmetic properties of certain polynomial sequences" [Bull. Amer. Math. Soc. **66** (1960), 202–204; MR **22** #1547].

F. Herzog (E. Lansing, Mich.)

7969:

Glénisson, Y.; Derwidué, L. Nouvelles généralisations des théorèmes de Fermat et de Wilson. Mathesis **69** (1960), 30–34.

Let $P(x)=x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial with rational integral coefficients and let S_k be the sum of the kth powers of its roots. Using the formula that expresses a_k in terms of the S_k, the authors derive congruence relations between various S_k which can be looked upon as being generalizations of Fermat's and Wilson's theorems. For example, if p is a prime $>n$, they prove that $(p-1)!S_p+S_1^p\equiv 0 \pmod p$; if $P(x)=x^{n-1}(x-1)$ this yields

Wilson's theorem and if $P(x)=x^{n-a}(x-1)^a$ it yields $a^p-a\equiv 0 \pmod p$, i.e., Fermat's theorem. Congruences with a prime power modulus are also derived, the following being typical of these: $S_{p^m}-S_{p^{m-1}}\equiv 0 \pmod{p^m}$.

H. W. Brinkmann (Swarthmore, Pa.)

7970:

Rokowska, B.; Schinzel, A. Sur un problème de M. Erdős. Elem. Math. **15** (1960), 84–85.

The reviewer asked: Does there exist a prime $p>5$ for which all the numbers $2!, 3!, \cdots, (p-1)!$ are incongruent mod p^2? The authors prove that such a prime (if it exists) must be $\equiv 5 \pmod 8$ and that for such a prime none of the residues $2!, \cdots, (p-1)!$ can be $\equiv -\frac{1}{2}(p-1)! \pmod p$. They also observe that no prime p between 5 and 1000 has the above property.

P. Erdős (Budapest)

7971:

Hartman, S.; Szüsz, P. On congruence classes of denominators of convergents. Acta Arith. **6** (1960), 179–184.

Verff. beweisen den folgenden Satz: (c_k) sei eine abnehmende Folge, $\sum_1^\infty c_k=\infty$. Sei $c_k'=c_k$ für $k\equiv b \pmod a$, $=0$ sonst. Dann ist für fast alle reellen Zahlen α für unendlich viele k die Ungleichung $|\alpha k-p|<c_k'$ mit passendem p, $(k,p)=1$, erfüllt. Der Beweis beruht auf einem Resultat von Duffin und Schaeffer [Duke Math. J. 8 (1941), 243–255; MR **3**, 71]. Einige weitere Anwendungen des Satzes werden gegeben.

H.-E. Richert (Göttingen)

7972:

Subba Rao, K. Some properties of Fibonacci numbers. II. Math. Student **27** (1959), 19–23.

[For part I see Bull. Calcutta Math. Soc. **46** (1954), 253–257; MR **17**, 238.] The author proves several theorems on Fibonacci and Lucas numbers, e.g., for every prime p there are infinitely many Lucas numbers $\equiv 1 \pmod p$. The Lucas numbers are defined as follows: $u_1=1$, $u_2=3$, $u_{n+1}=u_n+u_{n-1}$.

P. Erdős (Budapest)

7973:

Ferrier, A. Sur une propriété des nombres premiers $p=8k+1$. Mathesis **69** (1960), 141–142.

7974:

Carlitz, L. A determinant connected with Fermat's last theorem. Proc. Amer. Math. Soc. **11** (1960), 730–733.

Let $a_{ij}=\binom{p-1}{K}$ where p is a prime and $K\equiv j-i \pmod{p-1}$, and let Δ be the determinant of the matrix a_{ij}. This circulant is connected with the first case of Fermat's last theorem and Fermat's quotient $q(a)=(a^{p-1}-1)/p$. It is known that if $x^p+y^p+z^p=0$ is solvable in integers x, y, z not divisible by p, then $q(a)$ is divisible by p for all integers $a<47$. The author proves that in this case Δ is divisible by p^{p+43}. This result is significant when $p=6n+5$. For $p=6n+1$, $\Delta=0$. [See also Carlitz, same Proc. **10** (1959), 686–690; MR **21** #7182.]

D. H. Lehmer (Berkeley, Calif.)

7975:

Cassels, J. W. S. On a problem of Rankin about the Epstein zeta-function. Proc. Glasgow Math. Assoc. **4**, 73–80 (1959).

Let $h(m,n)=am^2+2bmn+cn^2$ be a positive definite quadratic form with determinant $ac-b^2=1$. A special form of this kind is $Q(m,n)=2\cdot 3^{-1/2}(m^2+mn+n^2)$. The author considers the Epstein zeta-function $Z_h(s)=\sum'\{h(m,n)\}^{-s}$ $(s>1)$, in which summation m, n assume all integral values and the prime indicates that the term with $m=n=0$ is excluded from the summation. For $s>1.035$, Rankin [same Proc. **1** (1953), 149–158; MR **15**, 507] proved that

$$(1) \qquad Z_h(s)-Z_Q(s)\geq 0$$

and that the sign of equality is needed only when h is equivalent to Q. Rankin asked if (1) is true for $s\geq 1$. (The function $Z_h(s)$ may be analytically continued over the whole plane, and its only singularity is at $s=1$.) The author proves that (1) holds for all $s\geq 0$.

S. Chowla (Boulder, Colo.)

7976:

Bašmakov, M. I.; Faddeev, D. K. Simultaneous representation of zero by a pair of quadratic quaternary forms. Vestnik Leningrad. Univ. **14** (1959), no. 19, 43–46. (Russian. English summary)

The authors give necessary and sufficient conditions in order that a pair of quaternary quadratic forms over a field k of characteristic $\neq 2$ or 3 be simultaneously equal to zero, and give an interpretation of the result as a condition for the simultaneous existence of rational points on a pair of curves in projective space over the field k.

W. H. Simons (Vancouver, B.C.)

7977:

Pitman, Jane. Davenport's constant for indefinite binary quadratic forms. Acta Arith. **6** (1960), 37–46.

Let $f(x,y)$ be an indefinite binary quadratic form with real coefficients and discriminant $D=D(f)$ and write $\Delta=\Delta(f)=\sqrt{D(f)}$. Let $P=(x',y')$ be any real point and $M(f;P)=\inf[|f(x+x', y+y')|;\ x, y \text{ integral}]$. Define $M(f)=\sup_P M(f;P)$ over all real points P. Davenport has shown the existence of a constant k such that, for all f, $M(f)>k\Delta(f)$. Thus an absolute constant K may be defined by $K=\sup[k; M(f)>k\Delta(f)]$, where the supremum is taken over all forms f.

Ennola showed that $K\geq 1/30.69$ and the author has shown that $K\leq 1/12$. This paper is an account of the theoretical basis for computations on EDSAC 2 at the Cambridge Mathematical Laboratory which establish the fact that $K\leq 1/12.921$.

B. W. Jones (Boulder, Colo.)

7978:

Mineev, M. P. A metric theorem on trigonometric sums with rapidly increasing functions. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 169–171. (Russian)

Let $g_0, g_1, \cdots$ be a sequence of natural numbers with $g_0=1$, and $g_j>1$ $(j>1)$. For natural numbers x, p write $F(x)=g_0g_1\cdots g_x$ and $S(\alpha,p)=\sum_{x=0}^{p-1}e^{2\pi i\alpha F(x)}$. Then the author shows that for any real $C>0$ the measure of the set of α in $0\leq\alpha\leq 1$ for which $|S(\alpha,p)|\leq Cp^{1/2}$ tends to

$1-e^{-C^s}$ as $p \to \infty$. In the proof the author first estimates the moments

$$\int_0^1 |S(\alpha, p)|^{2s}\, d\alpha,$$

using his earlier work on Tarry's problem [Mat. Sb. (N.S.) **46** (88) (1958), 451–454; MR **21** #2632], and then uses the general theory of the determination of probability distributions by moments.

J. W. S. Cassels (Cambridge, England)

7979:

Linnik, Yu. V. Some remarks on estimates of trigonometric sums. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 153–160. (Russian)

The author proves that, given a prime $p > n$, an integer $\lambda \geq 1$ and a set of n integers $N_1, N_2, \cdots, N_n$, the system of simultaneous congruences

$$x_1^r + x_2^r + \cdots + x_g^r \equiv N_r \pmod{p^\lambda} \quad (r = 1, 2, \cdots, n)$$

possesses a solution in integers $x_1, x_2, \cdots, x_g$ if $g \geq c_0 n^2 \log n$, where c_0 is a constant. The proof is based on I. M. Vinogradov's methods for trigonometric sums and A. Weil's famous inequality $|\sum_{x=0}^{p-1} \exp(2\pi i f(x)/p)| \leq (\nu-1)p^{1/2}$ (where f is a polynomial with integral coefficients and ν is the degree mod p of f) [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR **10**, 234]. The author remarks that if $p > c_1 n^2$ his method also leads readily to an asymptotic formula for the number of solutions. The special case of a homogeneous system ($N_1 = \cdots = N_n = 0$) possessing a non-trivial solution if $g \geq \frac{1}{2}n(n+1)+1$, follows, for $\lambda = 1$, from a theorem of C. Chevalley [Abh. Math. Sem. Univ. Hamburg **11** (1935), 73–75]. The result of this paper is relevant to certain problems in Diophantine analysis; for instance, to the 'singular series' associated with the application of the Hardy-Littlewood-Vinogradov method to the simultaneous solubility of the Diophantine equations $x_1^r + \cdots + x_g^r = N_r$ $(r = 1, 2, \cdots, n)$ [see K. K. Mardžanišvili, Mat. Sb. (N.S.) **33** (75) (1953), 630–675; MR **15**, 602].

H. Halberstam (London)

7980:

Putnam, C. R. On averages of the Riemann Zeta-Function. Arch. Math. **11** (1960), 346–349.

Let $A(t)$ and $B(t)$ denote, respectively, the real and imaginary parts of Riemann's zeta-function $\zeta(s)$, where $s = \sigma + it$. For $q = \text{const} > 0$, define $A_n = A(nq)$ and $B_n = B(nq)$ for $n = 0, 1, 2, \cdots$. Write $S_n = n^{-1} \sum_{k=1}^n B_k$, $T_n = n^{-1} \sum_{k=1}^n S_k$. The author obtains several results concerning B_n, S_n, T_n. We quote the following: (I) $\lim_{n\to\infty} B_n$ does not exist; (III) $\lim_{n\to\infty} T_n = 0$. It was previously shown by the author [Amer. J. Math. **76** (1954), 97–99; MR **15**, 412] that the sequence $B_1, B_2, \cdots$ does not have the limit 0.

S. Chowla (Boulder, Colo.)

7981:

Gel'fond, A. O. Some functional equations implied by equations of Riemann type. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 469–474. (Russian)

Verfasser beweist approximative Funktionalgleichungen für solche Dirichletreihen, die ein Potenzprodukt von $\zeta(s)$ und Dirichletschen L-Reihen sind. Die approximativen Formeln sind vom Riemann-Siegelschen Typus, d.h., durch Hinzunahme weiterer Terme kann die Genauigkeit beliebig weit getrieben werden. Speziell: Es sei $p \geq 1$, $\mu \geq 0$, $q > p/6 + \mu + 1$ (p, μ, q ganze Zahlen), $s = 1/2 + i\tau$, $1 \geq \alpha > \tau^{-1/2}$, $u(s) = [2(2\pi)^{-s}\Gamma(s) \cos \frac{1}{2}\pi s]^p$, $m = \frac{1}{2}u'(s)/u(s)$. Dann gilt

$$\begin{aligned}\zeta^p(s) &= \sum_{n \leq \exp(m-\alpha q)} \left[\nu_p(n) n^{-s} + \frac{\nu_p(n)}{u(s)} n^{s-1}\right] \\ &+ \sum_{\exp(m-\alpha q) < n \leq \exp(m+\alpha q)} \left[\nu_p(n) n^{-s} + \frac{\nu_p(n)}{u(s)} n^{s-1}\right] \psi_q\left(\frac{m - \ln n}{\alpha}\right) \\ &+ \sum_{\exp(m-\alpha q) < n \leq \exp(m+\alpha q)} \frac{1}{n^s} \sum^{\mu} \frac{c_k}{\alpha^k \tau^{k/2}} \psi_q^{(k)}\left(\frac{m - \ln n}{\alpha}\right) \\ &+ O((\alpha\sqrt{\tau})^{-\mu-1}\tau^{p/6}).\end{aligned}$$

Hierbei sind $\nu_p(n)$ die Lösungsanzahlen von $x_1 \cdots x_p = n$ in ganzen Zahlen x_k,

$$\psi_q(x) = \frac{1}{2\pi i} \int_{\sigma(>0)} \left(\frac{e^z - e^{-z}}{2z}\right)^q e^{xz} \frac{dz}{z}$$

und c_k gewisse Zahlen, die gleichmäßig in α beschränkt sind. Das entsprechende Restglied bei Dirichletschen L-Reihen wird gleichmäßig bezüglich des Moduls abgeschätzt. Druckfehler!

H.-E. Richert (Göttingen)

7982:

Rademacher, Hans. On the Phragmén-Lindelöf theorem and some applications. Math. Z. **72** (1959/60), 192–204.

In analytic number theory the Phragmén-Lindelöf theorem is usually applied to a half-strip, which is often less successful in cases where an infinity of functions is considered simultaneously, for then we need uniform estimates on the horizontal parts of the boundary of the (vertical) half-strip. The author presents theorems of Phragmén-Lindelöf type for the whole strip. The main result is as follows: Let $-Q < a < b$, put $s = \sigma + it$; let $f(s)$ be regular analytic in the strip $a \leq \sigma \leq b$, with $|f(s)| < C\exp(|t|^c)$ for suitable constants c and C; assume $|f(a+it)| \leq A|Q+a+it|^\alpha$, $|f(b+it)| \leq B|Q+b+it|^\beta$, where A, B, α, β are real constants, with $\alpha \geq \beta$; then we have

$$|f(s)| \leq (A|Q+s|^\alpha)^\lambda (B|Q+s|^\beta)^\mu$$

throughout the strip, with

$$\lambda = (b-\sigma)/(b-a), \quad \mu = (\sigma-a)/(b-a).$$

There are applications to the gamma function, e.g., that for $Q \geq 0$, $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$ we have

$$|\Gamma(\tfrac{1}{2}(Q+1-s))/\Gamma(\tfrac{1}{2}(Q+s))| \leq (\tfrac{1}{2}|Q+1+s|)^{\frac{1}{2}-\sigma}.$$

{The reviewer remarks that in the right-hand side $Q+1$ can be replaced by $Q' = \max(Q, 1-Q)$, an improvement indicated by the author for $Q \geq \frac{1}{2}$ only.}

A further application concerns an estimate for all Dirichlet L-functions $L(s, \chi)$ with primitive character χ mod k, uniformly with respect to k: If $0 < \eta \leq \frac{1}{2}$, $-\eta \leq \sigma \leq 1+\eta$, $k > 1$ we have

$$|L(s, \chi)| \leq (k|1+s|/2\pi)^{\frac{1}{2}(1+\eta-\sigma)}\zeta(1+\eta);$$

this is obtained from estimates on the line $\sigma = -\eta$, produced by the functional equation.

In a similar way the author derives estimates for the Dedekind ζ-function ζ_K. The result is somewhat simpler if K is normal over the rational field. In that case it reads: If $0 < \eta \leq \frac{1}{2}$, $-\eta \leq \sigma \leq 1+\eta$ we have

$$|\zeta_K(s)| \leq |\zeta(s)|(|d|(2\pi)^{1-n}|1+s|^{n-1})^{\frac{1}{2}(1+\eta-\sigma)}(\zeta(1+\eta))^{n-1},$$

where d is the discriminant and n is the degree of K.

There is a similar, but somewhat more complicated result for the Hecke functions $\zeta(s, \lambda)$ with "Grössencharacter" λ.

N. G. de Bruijn (Eindhoven)

7983:

Sierpiński, W. Sur un problème concernant les nombres $k \cdot 2^n + 1$. Elem. Math. **15** (1960), 73–74.

For each $k \leqq 100$, except perhaps 47 and 94, there is an n such that $k \cdot 2^n + 1$ is prime; see R. M. Robinson, Proc. Amer. Math. Soc. **9** (1958), 673–681 [MR **20** #3097]; Table 1. For $k=47$, $n \geqq 512$ if it exists. The author shows that there are no primes $k \cdot 2^n + 1$ where $k \equiv 1$ (mod $(2^{32}-1) \cdot 641$) and $k \equiv -1$ (mod $(2^{32}+1)/641$). The proof is easy. The problem of finding the smallest k with $k \cdot 2^n + 1$ always composite remains open. If for some k and some P, $(2^n + k, P) > 1$ for all n, then $(k \cdot 2^n + 1, P) > 1$ for all n. This does not imply that $k \cdot 2^n + 1$ is composite for all n, as is asserted in this note.

J. L. Selfridge (Seattle, Wash.)

7984:

Inkeri, K. Tests for primality. Ann. Acad. Sci. Fenn. Ser. A I No. 279 (1960), 19 pp.

In the first part of this paper the author uses the theory of quadratic fields to develop several primality tests for numbers of the form $h \cdot 2^n - 1$. The following theorem is representative. Let $N = h \cdot 2^n - 1$ be a number satisfying the conditions $n \geqq 3$, $h < 6 \cdot 2^n - 5$ when $h \equiv \pm 1$ (mod 6), $h < 2^{n+1} - 1$ when $h \equiv 3$ (mod 6). Let p be a prime such that $p \equiv 1$ (mod 4) and $(N|p) = -1$. Let α be a primitive unit in the field $k(\sqrt{p})$ and put $\beta = \alpha^2$, $v_0 = \beta^h + \beta^{-h}$, $v_i = v_{i-1}^2 - 2$ $(i = 1, 2, \cdots)$. Then a necessary and sufficient condition that N be prime is that $v_{n-2} \equiv 0$ (mod N). The author's theorems include as special cases Lucas' well-known test for the primality of Mersenne's numbers and related theorems of Lehmer [Ann. of Math. (2) **31** (1930), 419–448], Brewer [Duke Math. J. **18** (1951), 757–763; MR **13**, 208] and Riesel [Ark. Mat. **3** (1956), 245–253; MR **17**, 945]. In the second part of the paper the author derives primality tests for numbers of the form $h \cdot 2^n + 1$. Although the natural basis for testing such numbers is the converse of Fermat's theorem the author uses Lucas' sequences instead. For example, he obtains the test: Fermat's number $F_m = 2^{2^m} + 1$ $(m \geqq 2)$ is prime if and only if F_m divides the term v_{2^m-2} of the series $v_0 = 8$, $v_i = v_{i-1}^2 - 2$ $(i = 1, 2, \cdots)$.

A. L. Whiteman (Princeton, N.J.)

7985:

Levin, B. V. Estimates from below for the number of nearly-prime integers belonging to some general sequences. Vestnik Leningrad. Univ. **15** (1960), no. 7, 48–65. (Russian. English summary)

It is not known whether the polynomial sequence $\{n^2+1\}$ contains infinitely many primes; the result is probably true, but a proof seems out of the reach of existing methods. The author proves the corresponding simpler problem for almost-primes. He proves that: The sequence $\{n^2+1\}$ $(n = 1, 2, \cdots, N)$ contains at least $aN/(\log N) + O(N \log \log N/(\log N)^{3/2})$ members each having at most five prime factors. The prime factors exceed $N^{1/2.91}$ and a is a certain positive constant. The author bases his proof on A. I. Vinogradov's account of A. Selberg's 'lower bound' sieve method [Mat. Sb. (N.S.) **41** (**83**) (1957), 49–80, 415–416; MR **20** #3836]. The paper ends with a generalisation.

H. Halberstam (London)

7986:

Lavrik, A. F. Estimation of certain integrals connected with additive problems. Vestnik Leningrad. Univ. **14** (1959), no. 19, 5–12. (Russian. English summary)

Let p denote a prime and let γ, δ be a pair of non-negative numbers satisfying $0 < \delta - \gamma \leqq 1$. Define

$$S(\vartheta, N) = \sum_{p \leq N} \exp(2\pi i \vartheta p),$$

$$S_1(\vartheta, N) = \sum_{n=1}^{\infty} \Lambda(n) \exp\left(-\frac{n}{N} + 2\pi i \vartheta n\right),$$

$$T(\vartheta, N) = \sum_{1 \leq n^2 \leq N} \exp(2\pi i \vartheta n^2);$$

as usual, $\Lambda(n)$ denotes von Mangoldt's function. The author proves that

$$\int_\gamma^\delta |S(\vartheta, N)|^2 \, d\vartheta = (\delta - \gamma)(N/\log N) + O(N(\log \log N/\log N)^2),$$

$$\int_\gamma^\delta |S_1(\vartheta, N)|^2 \, d\vartheta = \tfrac{1}{2}(\delta - \gamma)N \log N + O(N(\log \log N)^2),$$

$$\int_\gamma^\delta |T(\vartheta, N)|^2 \, d\vartheta = (\delta - \gamma)N^{1/2} + O((\log N)^2).$$

Such results are of interest in the application of the circle method to additive problems in which the minor arcs cannot be dealt with by the usual method of estimating trigonometric sums [see Yu. V. Linnik, Mat. Sb. (N.S.) **32** (74) (1953), 3–60; MR **15**, 602]. The author uses Fourier series to pick out the interval (γ, δ) from $(0, 1)$, in the manner described in Lemma 12 of I. M. Vinogradov's book on trigonometric sums [*The method of trigonometrical sums in the theory of numbers*, Trav. Inst. Math. Stekloff **23** (1947); MR **10**, 599], and the Viggo Brun method.

H. Halberstam (London)

7987:

Satyanarayana, U. V. On the sequence $\{V_n\}$, $V_n = \sum_{1 \leq i \leq k} a_i V_{n-i}$, $k \geqq 2$. Math. Student **27** (1959), 37–42.

The author generalizes several of the inequalities proved by Subba Rao [Amer. Math. Monthly **60** (1953), 680–684; MR **15**, 401] to sequences defined by more general recursion formulas, and obtains some new inequalities.

P. Erdös (Budapest)

7988:

Vološin, Yu. I. [Vološins, J. M.] On the integral part of a rational number. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), no. 4, 95–98. (Russian. Latvian summary)

For fixed n, $[m/n]$ is considered as a function of m, where [ ] denotes the greatest integer function. It is shown that

$$[m/n] = \frac{m}{n} - \frac{n-1}{2n} - \frac{1}{n}\sum_{k=1}^{n-1} \varepsilon^{mk} \frac{\varepsilon^k}{1-\varepsilon^k},$$

where $\varepsilon = \exp(2\pi i/n)$.

W. H. Simons (Vancouver, B.C.)

7989:

Palamà, Giuseppe. Su di una congettura di Schinzel. Boll. Un. Mat. Ital. (3) 14 (1959), 82–94.

Let x_i be positive integers and let A_s be positive rationals m/n of the form

$$\frac{1}{x_1} \pm \frac{1}{x_2} \pm \cdots \pm \frac{1}{x_s},$$

where the signs + and − may be chosen arbitrarily. It has been conjectured by Andrzej Schinzel that every positive rational number m/n, with n greater than a certain natural number k_m, dependent on m, is an A_3 [W. Sierpiński, Mathesis 65 (1956), 16–32; MR 17, 1185]. This conjecture has been verified for all $m<20$. For some m's this proof is quite complicated. For example, Schinzel has verified that for $n>23$, the number $18/n$ is an A_3. In the present paper the author proves a result [Th. III, p. 92] from which follows readily the conjecture of Schinzel for $m=19$, 20, 21, 22, 23. *I. A. Barnett* (Cincinnati, Ohio)

7990:

Yin, Wen-lin. Remarks on the representation of large integers as sums of primes. Acta Sci. Nat. Univ. Pekinensis No. 3 (1956), 323–326. (Chinese. English summary)

This is the original Chinese version of the article reviewed in MR 19, 16 [Bull. Acad. Polon. Sci. Cl. III 4 (1956), 793–795]. *C.-S. Lin* (Rizal)

7991:

Makowski, A. Partitions into unequal primes. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 8 (1960), 125–126. (Russian summary, unbound insert)

It was proved by H.-E. Richert [Norsk Mat. Tidsskr. 31 (1949), 120–122; Math. Z. 52 (1949), 342–343; MR 11, 646, 502] that every integer >6 can be represented as a sum of unequal primes. Analogous results are presented here: Every integer >55 is a sum of unequal primes of the form $4k-1$, and similar theorems hold for primes of the form $4k+1$, or $6k-1$, or $6k+1$, for integers exceeding 121, 161, 205, respectively. *N. G. de Bruijn* (Eindhoven)

7992:

Val'fiš, A. A. Representation of numbers as sums of generalized pentagonal numbers. Soobšč. Akad. Nauk Gruzin. SSR 22 (1959), 385–392. (Russian)

A generalised pentagonal number $t_5(x)$ is defined, for integral x, by the relation $t_5(x)=\frac{3}{2}x^2-\frac{1}{2}x$. Let $r_s(n)$ denote the number of representations of a positive integer n in the form $n=\sum_{k=1}^{s} t_5(x_k)$; in other words, $r_s(n)$ is the number of representations of $M=24n+s$ as the sum of s squares of integers, each integer congruent to -1 (mod 6). Exact formulae for $r_s(n)$ have been computed when $s=3$ by H. Streefkerk [Thesis, Free University of Amsterdam, 1943; MR 7, 414], and, when $s=4$, 5, 6 and 7, by G. A. Lomadze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 22 (1956), 77–102; MR 19, 942]. The author now employs a similar analysis to accomplish the same for $s=8$. *H. Halberstam* (London)

7993:

Gupta, Hansraj. Graphic representation of a partition of a j-partite number. Res. Bull. Panjab Univ. (N.S.) 10 (1959), 189–196.

The author begins by giving a method of arranging the partitions of a j-partite number (i.e., a j-dimensional vector $N_j=(n_1, \cdots, n_j)$ with positive integral components) in "ascending" or "descending" order. For $j=2$ the parts are represented as vectors arranged in order of increasing slope, or increasing magnitude if there are two or more vectors with the same slope. There is an obvious extension to j dimensions, $j>2$. The author then obtains a number of results such as

$$k'\cdot q(N_j, k) \leqq \prod_{i=1}^{j} \binom{n_i-1}{k-1} \leqq k'\cdot p(N_j, k),$$

where $p(N_j, k)$ denotes the number of partitions of N_j into exactly k parts, and $q(N_j, k)$ the number into exactly k distinct parts.

In spite of the title of the note, very little use is made of the given graphical representation of a j-partite number, and the proofs involve only elementary algebra. *R. D. James* (Vancouver, B.C.)

7994:

Schmidt, Wolfgang. On normal numbers. Pacific J. Math. 10 (1960), 661–672.

Let r and s be two integers greater than 1. Under what circumstances does normality of a real number ξ, $0<\xi<1$, in the scale of r imply normality of ξ in the scale of s? (Normality means that, for all $k\geqq 1$, each combination of k digits occurs with the proper frequency.) Let the notation $r\sim s$ stand for the fact that r and s are powers of the same integer. It is fairly obvious that, in the case $r\sim s$, normality of ξ in the scales of r and s imply each other. (The paper contains a formal proof of this fact.) If $r\nsim s$, then this implication does not hold. In fact, the author shows that in this case there are c (power of the continuum) numbers ξ which are normal in the scale of r but not even simply normal in the scale of s. (Simple normality means that each single digit occurs with the proper frequency.) He derives this result from the following theorem. Let $1<t<s$; then almost all numbers $\xi=\sum a_n t^{-n}$, $0<\xi<1$, written in the scale of t, have the property that the corresponding number $\eta=\sum a_n s^{-n}$ (which is obviously not simply normal in the scale of s) is normal in every scale r for which $r\nsim s$. One of the principal tools used in the proof of this theorem is the theory of uniform distribution modulo 1. *F. Herzog* (E. Lansing, Mich.)

7995:

Kolberg, O. Note on the parity of the partition function. Math. Scand. 7 (1959), 377–378.

Congruence properties of the partition function $p(n)$ have been the subject of numerous investigations for over forty years; but the knowledge gained in this field, though substantial, is still fragmentary. In the present paper the author uses Euler's identity

$$\sum (-1)^k p\{n-\tfrac{1}{2}k(3k\pm 1)\} = 0$$

to deduce that $p(n)$ takes both even and odd values infinitely often. Although the argument is extremely short and simple, this theorem on the parity of $p(n)$ appears to

have been overlooked by earlier writers. A combination of this result with well-known identities of Jacobi and Ramanujan leads to the conclusion that each of the congruences $p(n)\equiv 0 \pmod{10}$, $p(n)\equiv 5 \pmod{10}$, $p(n)\equiv 0 \pmod{14}$, $p(n)\equiv 7 \pmod{14}$ is satisfied for infinitely many values of n.
L. Mirsky (Sheffield)

7996:

Hlawka, Edmund. Über C-Gleichverteilung. Ann. Mat. Pura Appl. (4) **49** (1960), 311–325.

Let $a(t, T)$ $(t\geqq 0;\ T\geqq 0)$ be $\geqq 0$ with $\int_0^T a(t, T)\,dt=1$. If $\chi(x;\ \alpha, \beta)$ is the characteristic function of the interval $\alpha\leqq x<\beta$, with $0\leqq\alpha<\beta\leqq 1$, if $\{u\}=u-[u]$, and if finally $f(t)$ $(t\geqq 0)$ is integrable, then let

$$D = D(A, f, T, \alpha, \beta) = \int_0^T a(t, T)\,\chi(\{f(t)\}; \alpha, \beta)\,dt-(\beta-\alpha),$$

where A denotes the integral operator $\int_0^T a(t, T)$. The function $f(t)$ is said to have a uniform A-distribution if, for each α and β with $0\leqq\alpha<\beta\leqq 1$, $D\to 0$ as $T\to\infty$; this relation holds then uniformly in α and β. In the special case $A(t, T)=T^{-1}$ the distribution is called a C-distribution. These distributions have already been treated by L. Kuipers and B. Meulenbeld [Nederl. Akad. Wetensch. Proc. Ser. A **52** (1949), 1151–1157, 1158–1163; **53** (1950), 226–232, 305–308; **56** (1953), 340–348; MR **11**, 423, 424, 648; **15**, 410]. The supremum $D(A, f, T)$ of D for $0\leqq\alpha<\beta\leqq 1$ is called the A-discrepancy of f. A function f has a uniform A-distribution if and only if $D(A, f, T)\to 0$ as $T\to\infty$. For each integer $g\neq 0$, $D(A, gf, T)\leqq 3|g|D(A, f, T)$; therefore, if f has a uniform A-distribution, then also gf has. The author gives an upper bound for the absolute value of $D(A, f, T)-D(A, g, T)$ under the assumption that an upper bound for the absolute value of $f-g$ is known. Two functions with a constant difference have the same discrepancy. If $w(t)$ is a function of t $(0\leqq t\leqq 1)$ with bounded variation V, then

$$\left|\int_0^T a(t, T)w(\{f(t)\})\,dt-\int_0^1 w(t)\,dt\right| \leqq VD(A, f, T).$$

Under general conditions the author obtains upper bounds for the absolute value of $D(A, f, T)$, with applications to uniform distribution, for instance: If there exists a $\sigma>0$ such that $g(t+\sigma)-g(t)$ tends, for $t\to\infty$, monotonically decreasing to zero, with $t|g(t+\sigma)-g(t)|\to\infty$, then $g(t)$ has a uniform C-distribution. Under the assumption that an upper bound is known for the absolute value of $D(C, \{g(t+h\sigma)-g(t)\}, T-|h|\sigma)$, the author finds an upper bound for the absolute value of $D(C, \{g\}, T)$. This result, which implies for instance that $g(t)$ has a uniform C-distribution if this is the case with $g(t+h\sigma)-g(t)$ for each positive integer h, is a generalisation of the corresponding result obtained by Van der Corput in the theory of discrete uniform distributions, where t traverses the sequence of the positive integers [Van der Corput and Ch. Pisot, Nederl. Akad. Wetensch. Proc. **42** (1939), 476–486, 554–565, 713–722; MR **1**, 66].
J. G. van der Corput (Berkeley, Calif.)

7997:

Carlitz, L. A note on Gauss' first proof of the quadratic reciprocity theorem. Proc. Amer. Math. Soc. **11** (1960), 563–565.

The author gives a simple version of Gauss' first proof of the quadratic reciprocity law.
W. H. Mills (Berkeley, Calif.)

7998:

Godwin, H. J. The determination of units in totally real cubic fields. Proc. Cambridge Philos. Soc. **56** (1960), 318–321.

Let ε_1, ε_2 be independent units of a totally real cubic field. For any unit ε of the field, with conjugates ε', ε'', define $2S(\varepsilon)=(\varepsilon-\varepsilon')^2+(\varepsilon'-\varepsilon'')^2+(\varepsilon''-\varepsilon)^2$. The author proves that if $S(\varepsilon)>34$ and $S(\varepsilon_2)\geqq 122$, then either ε_1, ε_2 are a pair of fundamental units, or there is a unit η such that $\eta=\varepsilon_1^{1/2}\varepsilon_2^{1/2}$ and $S(\eta)<\{(81/2)S(\varepsilon_1)S(\varepsilon_2)\}^{1/2}$, or $\eta=\varepsilon_1^{2/3}\varepsilon_2^{1/3}$ and $S(\eta)<\{(243/2)S(\varepsilon_1^2)S(\varepsilon_2)\}^{1/3}$. The author also conjectures that if $S(\varepsilon)>9$ then ε_1, ε_2 are a pair of fundamental units.
M. Newman (Washington, D.C.)

7999:

Rédei, Ladislaus. Natürliche Basen des Kreisteilungskörpers. II. Abh. Math. Sem. Univ. Hamburg **24** (1960), 12–40.

This part consists of the proof of the most difficult result of part I [same Abh. **23** (1959), 180–200; MR **21** #2644], viz., the one concerning "balanced" sets when n is the product of three different primes.
N. G. de Bruijn (Eindhoven)

8000:

Uchiyama, Saburô. On the Thue-Siegel-Roth theorem. I, II. Proc. Japan Acad. **35** (1959), 413–416, 525–529.

In part I, the T-S-R theorem is refined in the case of an imaginary quadratic field K, as follows: If α is a nonzero algebraic number, and $\kappa>1$ is fixed, then the inequality $|\alpha-\xi|<(M(\xi))^{-\kappa}$ has only finitely many nonrational solutions ξ in K. Here $M(\xi)$ is the absolute value of the coefficient of x^2 in the polynomial with coprime rational integral coefficients which defines ξ. A corollary is that for $\nu>2$, the inequality $0<|\alpha-p/q|<|q|^{-\nu}$ has only finitely many integral solutions p, q in K.

In part II, statements are given of generalizations to the case of algebraic approximants of related theorems by Ridout [Mathematika **4** (1957), 125–131; MR **20** #32] and Mahler [ibid. 122–124; MR **20** #33]. Proof of the following theorem is also given: Let $\alpha\neq 0$ be algebraic, and let K be an algebraic number field. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be a finite set of prime ideals with distinct rational primes $p(\mathfrak{p}_1), \cdots, p(\mathfrak{p}_s)$ in an arbitrary finite extension L of $K(\alpha)$. Then for each $\kappa>2$ the inequality

$$\prod_{k=1}^{s} |\alpha-\xi|_{\mathfrak{p}_k} < (H(\xi))^{-\kappa}$$

has only finitely many solutions ξ in K. Here $H(\xi)$ is the maximum of the absolute values of the coefficients in the polynomial with coprime rational integral coefficients which defines ξ.
W. J. LeVeque (Ann Arbor, Mich.)

8001:

Uchiyama, Saburô. On the Thue-Siegel-Roth theorem. III. Proc. Japan Acad. **36** (1960), 1–2.

Let k be a field of characteristic 0, t an indeterminate, and $k\langle t\rangle$ the field of all formal series $\alpha=\sum_{-\infty}^{f} c_n t^n$, where the c_n lie in k and f is finite. If $c_f\neq 0$, put $|\alpha|=e^f$; further

put $|0|=0$; then $|\alpha|$ is a valuation on $k\langle t\rangle$. $k\langle t\rangle$ contains the rational function field $k(t)$ as a subfield. In analogy to K. F. Roth's theorem [Mathematika **2** (1955), 1–20; corrigendum, 168; MR **17**, 242], the author notes that if $\alpha \in k\langle t\rangle$ is algebraic over $k(t)$, the inequality

$$0 < |\alpha - p/q| < q^{-\kappa} \quad (\kappa > 2)$$

has only finitely many solutions in polynomials p, $q \neq 0$. The same result was already proved by D. Fenna in his Manchester Ph.D. thesis of 1956.

K. Mahler (Manchester)

8002:

Cugiani, Marco. Sull'approssimabilità di un numero algebrico mediante numeri algebrici di un corpo assegnato. Boll. Un. Mat. Ital. (3) **14** (1959), 151–162. (English summary)

The following theorem is proved. Let K be an algebraic number field and let α be an algebraic number of degree $g \geq 2$ over K. Let $\varepsilon > 0$ and let $f(q)$ denote the function

$$f(q) = (8g+\varepsilon)(\log\log\log q)^{-1/2}.$$

If $\{\xi_i\}$ is a sequence of primitive numbers of K such that $|\alpha-\xi_i| < q_i^{-(2+f(q_i))}$, where q_i denotes the height of ξ_i, then we have

$$\limsup_{i\to\infty} \frac{\log q_{i+1}}{\log q_i} = +\infty.$$

The author's proof makes extensive use of the method of Roth-LeVeque, as exposed in W. J. LeVeque, *Topics in number theory* [Addison-Wesley, Reading, Mass., 1956; MR **18**, 283; Vol. 2, Ch. 4].

C. G. Lekkerkerker (Amsterdam)

8003:

Steuerwald, Rudolf. Über unendliche C-Kettenbrüche, deren korrespondierende Potenzreihe einer quadratischen Gleichung genügt. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 219–250 (1960).

Let $P(x) = 1 + c_1x + c_2x^2 + \cdots$ be a function which is not reducible to a rational function, but which satisfies a quadratic equation. Let it be expanded into an infinite C-fraction by means of the equations

$$P_\lambda(x) = a_{\lambda+1}\frac{x^{r_{\lambda+1}}}{P_{\lambda+1}(x)}, \quad \lambda = 0, 1, \cdots, \quad a_{\lambda+1} \neq 0, \quad r_{\lambda+1} > 0.$$

This paper studies the analogy between such continued fractions and regular periodic continued fraction expansions of quadratic irrational numbers.

E. Frank (Chicago, Ill.)

8004:

Schmidt, Wolfgang. A metrical theorem in geometry of numbers. Trans. Amer. Math. Soc. **95** (1960), 516–529.

Let S be a Borel set in the n-dimensional space R_n not containing the origin. Let $V = V(S)$ be its volume, $L(S)$ the number of lattice points g (i.e., points g with integral coordinates) in S, and $P(S)$ the number of primitive lattice points in S. The author considers the discrepancies $D(S) = |L(S)V(S)^{-1}-1|$, $E(S) = |P(S)\zeta(n)V(S)^{-1}-1|$. His first theorem runs as follows.

Let $\psi(s)$ $(s \geq 0)$ be a positive, nondecreasing function with $\int_0^\infty \psi(s)^{-1}\,ds < \infty$. Suppose $n \geq 3$. Then, for almost all $n\times n$-matrices A,

$$D(AS) = O(V^{-1/2}\log V\psi^{1/2}(\log V)),$$
$$E(AS) = O(V^{-1/2}\log V\psi^{1/2}(\log V)),$$

if S runs through any family Φ which is totally ordered (by inclusion), such that all $V(S)$ are finite and $V(S)\to\infty$ if S runs through Φ.

The proof uses integration of expressions like

$$\left(\sum_g \rho(Ag) - V(S)\right)^2$$

over certain fundamental regions of matrices, ρ denoting the characteristic function of S. Results of the above type are not true for $n=1$. A slightly weaker result is proved in the case $n=2$, by the same method. But here the details are essentially more complicated.

C. G. Lekkerkerker (Amsterdam)

FIELDS

See also 8001.

8005:

Carlitz, L. A theorem on permutations in a finite field. Proc. Amer. Math. Soc. **11** (1960), 456–459.

Let Fq be the field with $q=p^n$ elements where p is odd. The quadratic character $\chi(x)$ is 0, $+1$, -1 according as x is 0, a nonzero square, or a non-square in Fq. A question raised by W. F. Pierce is the following: Find the polynomial functions $f(x)$ over Fq such that (1) $f(0)=0$, $f(1)=1$ and (2) $\psi(f(a)-f(b)) = \psi(a-b)$, all $a, b \in Fq$. It is shown here that necessarily $f(x) = x^{p^j}$ for some j in the range $0 \leq j < n$. In a letter to MR, the author remarks that condition (2) implies directly that $f(x)$ is a permutation polynomial, and this is therefore a redundant condition in his hypothesis! This result is somewhat negative in its applications to the theory of projective planes since it shows that a construction by Pierce, analogous to the Moulton construction of non-Desarguesian planes, can yield only Desarguesian planes in the finite case.

Marshall Hall, Jr. (Pasadena, Calif.)

8006:

Popovič, Konstantin P. [Popovici, Constantin P.] Unique factorization rings of Dirichlet integers. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 441–448.

In the Dirichlet relative-quadratic field $R(i, \delta^{1/2})$, let $\delta = \pi_1\pi_2\cdots\pi_t$ be a square-free Gaussian integer factored into primes. Then if unique factorization were to hold for indecomposables, either δ is prime (or i) or else $(1+i)\nmid\delta$, $\delta \not\equiv 1, 3 \pmod 4$ and $t=2$. The method consists essentially in showing that $\delta^{1/2}$ would then have to be an indecomposable.

H. Cohn (Tucson, Ariz.)

8007:

Faith, Carl. Correction to "Galois extensions". Proc. Amer. Math. Soc. **11** (1960), 670.

The article being corrected appeared in same Proc. **9** (1958), 222–229 [MR **20** #2357].

8008:

Lamprecht, Erich. Bewertungssysteme und Stellenringe algebraischer Funktionenkörper. Abh. Math. Sem. Univ. Hamburg **24** (1960), 91–108.

Let R be a normal affine domain of transcendence degree n over a field k and assume that k is maximally algebraic in the quotient field A of R. An ordered n-tuple $(K_1, K_2, \cdots, K_n)$ of fields is said to be a structural representation of A relative to R when: $k \subset K_i \subset A$; transcendence degree $(K_i/k) = 1$; K_i is maximally algebraic in A; $K_1, \cdots, K_n$ are algebraically independent over k; $R \cap K_i$ is integrally closed and has K_i as quotient field. To each such structural representation of A there corresponds a unique system $B(K_1, K_2, \cdots, K_n)$ of discrete rank n valuations of A/k. Roughly speaking, the construction of a member of B is this: Start with a prime divisor $\mathfrak{Z}_1$ of K_1/k, extend it to a prime divisor $\mathfrak{P}_1$ of A/k, apply the natural mapping associated with $\mathfrak{P}_1$ to the fields $A, K_2, \cdots, K_n$, and repeat the process on the image fields. A composition then yields a discrete rank n valuation of A. With some minor restrictions on A, k, and R, the author describes a construction that associates a unique member of $B(K_1, K_2, \cdots, K_n)$ to each local ring $\mathfrak{O}$ of a subset V of the affine model M defined by R. The subset V consists of almost all the local rings of maximal dimension in M. The construction is an inductive one and the main burden of the proof is to show that the necessary properties are preserved when everything is subjected to the mapping defined by a prime divisor. The effect of a permutation of the structural representation upon this construction is also investigated.

H. T. Muhly (Iowa City, Iowa)

8009:

Yakabe, Iwao. On divisors in the field of algebraic functions of two variables. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 146–151.

Es sei A ein endlich-algebraischer Funktionenkörper vom Transzendenzgrad 2 über dem in A algebraisch abgeschlossenen Konstantenkörper k. Eine "Erzeugung" von A ist nach Lamprecht definiert als ein Paar über k unabhängiger Teilkörper K, K' vom Transzendenzgrad 1, von denen jeder in A algebraisch abgeschlossen ist. Dazu gehört in A ein System $B = B(K, K')$ von einrangigen Primdivisoren $\mathfrak{P}$ von A, definiert dadurch, dass $\mathfrak{P}$ in K' aber nicht in K trivial ist. Ist $\mathfrak{P} \in B$, und ist $\mathfrak{q}$ ein Primdivisor des Restkörpers $A\mathfrak{P}$ über $K\mathfrak{P}$, so liefert die Hintereinanderausführung der Restabbildungen mod $\mathfrak{P}$ und mod $\mathfrak{q}$ einen Ort von A, dessen Bewertung diskret und vom Range 2 ist; er wird mit $(\mathfrak{P}, \mathfrak{q})$ bezeichnet. Es sei $b = b(K, K')$ das System dieser $(\mathfrak{P}, \mathfrak{q})$. Nach dem Vorbild von Lamprecht werden in Bezug auf dieses Bewertungssystem Divisoren gebildet. Zu jedem solchen b-Divisor $\mathfrak{A}$ kann in üblicher Weise der k-Vektorraum $\mathfrak{M}(\mathfrak{A})$ der Vielfachen von $\mathfrak{A}$ in A gebildet werden. Ziel des Verf. ist der Beweis des folgenden Endlichkeitssatzes: Es sei T eine endliche Menge von Primdivisoren $\mathfrak{P} \in B$. Falls der Divisor $\mathfrak{B}$ ein "starkes Vielfaches von $\mathfrak{A}$ auf T" ist, so ist $M(\mathfrak{A})/M(\mathfrak{B})$ endlichdimensional, und zwar ist die Dimension $\leq \sum_{\mathfrak{P} \in T} (d_{\mathfrak{P}}(\mathfrak{B}) - d_{\mathfrak{P}}(\mathfrak{A}))$, wobei $d_{\mathfrak{P}}(\mathfrak{A})$ den Grad des Restdivisors von $\mathfrak{A}$ modulo $\mathfrak{P}$ bedeutet. Der hier eingehende Begriff des "starken Vielfachen" wird wie folgt gefaßt: Ein b-Divisor $\mathfrak{A}$ ist eine formale Summe

$$\sum_{(\mathfrak{P}, \mathfrak{q}) \in b} (a_{\mathfrak{P}}, a_{\mathfrak{q}})(\mathfrak{P}, \mathfrak{q}),$$

wobei $(a_{\mathfrak{P}}, a_{\mathfrak{q}})$ ein Paar ganzrationaler Zahlen ist (also ein Element der Wertgruppe von $(\mathfrak{P}, \mathfrak{q})$), und wobei nach Lamprecht gewisse Homogenitäts- und Endlichkeitsbedingungen erfüllt sind. Ein zweiter Divisor $\mathfrak{B}$ heißt Vielfaches von $\mathfrak{A}$, wenn $(b_{\mathfrak{P}}, b_{\mathfrak{q}}) \geq (a_{\mathfrak{P}}, a_{\mathfrak{q}})$. Er heißt "starkes" Vielfaches von $\mathfrak{A}$, wenn außerdem $b_{\mathfrak{P}} = a_{\mathfrak{P}}$ für alle $\mathfrak{P}$. Er heißt "starkes Vielfaches auf T" wenn außerdem $b_{\mathfrak{q}} = a_{\mathfrak{q}}$ für alle $\mathfrak{q}$, die zu einem $\mathfrak{P} \notin T$ gehören. [Literatur: Lamprecht, Math. Ann. **132** (1957), 373–403; MR **19**, 244].

P. Roquette (Tübingen)

8010:

Cheo, Peh-Hsuin. On primal rings. Acta Math. Sinica **6** (1956), 542–547. (Chinese. English summary)

Let R be a commutative ring with unity and no zero divisor. The author proves four theorems concerning necessary and sufficient conditions for R to be a primal ring. Theorem 1: R is a valuation ring of the quotient field K of R, if and only if R is primal and every non-principal ideal cannot be generated by a finite number of elements. Theorem 2: R is a subring of a valuation ring Q of the quotient field K of R, and the set of all non-units of R coincides with the maximum prime ideal P of Q, if and only if (1) R is primal, (2) whenever $a, b \in R$, $a \nmid b$ and $b \nmid a$, the quotients $(a)b^{-1}$ and $(b)a^{-1}$ always equal the maximum prime ideal P of R. Theorem 3: R is primal if and only if the set of all prime ideals of R is simply ordered by inclusion. Theorem 4: R (which stands for a unique factorization ring) is primal if and only if it is a valuation ring of its quotient field with respect to some discrete archimedean valuation ring.

Ching-ju Chang (Taipei)

ABSTRACT ALGEBRAIC GEOMETRY

See also 7976, 8008.

8011:

Nobusawa, Nobuo. A note on algebraic theory of algebraic function fields with several variables. Arch. Math. **11** (1960), 183–187.

Es sei K ein algebraischer Funktionenkörper in n Variablen über einem beliebigen Konstantenkörper k, und es seien $x_1, \cdots, x_n$ algebraisch unabhängige Elemente in K über k. Es sei S die Menge derjenigen Primdivisoren (im Sinne der Bewertungstheorie) von K, welche mindestens einen der Körper $k(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ identisch bewerten. S_1 bedeute die Teilmenge derjenigen Primdivisoren aus S, welche $k(x_1)$ nicht identisch bewerten, welche also aus einem Primdivisor von $k(x_1)/k$ durch Funktionalerweiterung bezüglich $(x_2, \cdots, x_n)$ entstehen. Man setze $S_2 = S - S_1$. Es sei $\mathfrak{a}$ ein Divisor von K, welcher sich nur aus Primdivisoren aus S_1 zusammensetzt; die Vielfachheit eines Primdivisors $\mathfrak{p}$ in $\mathfrak{a}$ werde mit $w_{\mathfrak{p}}(\mathfrak{a})$ bezeichnet. Entsprechend sei $\mathfrak{b}$ ein Divisor von K, welcher sich nur aus Primdivisoren aus S_2 zusammensetzt; Vielfachheiten $w_{\mathfrak{p}}(\mathfrak{b})$. Man bilde den Modul $R(\mathfrak{a}, \mathfrak{b})$ derjenigen Elemente $a \in K$, für welche $w_{\mathfrak{p}}(a) \geq w_{\mathfrak{p}}(\mathfrak{a})$ für $\mathfrak{p} \in S_1$ und $w_{\mathfrak{p}}(a) \geq w_{\mathfrak{p}}(\mathfrak{b})$ für $\mathfrak{p} \in S_2$; hierbei bedeutet $w_{\mathfrak{p}}$ die zu $\mathfrak{p}$ gehörige normierte Bewertung von K. Ziel dieser Note ist es erstens, zu zeigen daß $R(\mathfrak{a}, \mathfrak{b})$ endlichdimensional ist, und zweitens, eine Art Riemann-Rochschen Satz für Moduln dieser Art zu beweisen.

P. Roquette (Tübingen)

8012:

Shimura, Goro. Fonctions automorphes et correspondances modulaires. Proc. Internat. Congress Math. 1958, pp. 330–338. Cambridge Univ. Press, New York, 1960.

A brief exposition of the main results achieved by the author in his two papers in Ann. of Math. (2) **70** (1959), 101–144 and in J. Math. Soc. Japan **10** (1958), 1–28 [MR **21** #6370; **20** #1679]. *A. Weil* (Princeton, N.J.)

8013:

Matsusaka, Teruhisa. A correction to "On a characterization of a Jacobian variety". Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 350.

The paper here corrected appeared in the same Mem. **32** (1959), 1–19 [MR **21** #7213].

8014:

Abe, Eiichi. On the automorphisms of some simple groups. Proc. Japan Acad. **34** (1958), 315–318.

Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field of characteristic 0. Let $A(\mathfrak{g})$ be the (linear algebraic) group of all automorphisms of $\mathfrak{g}$, G the irreducible identity component of $A(\mathfrak{g})$. The object of the paper is to determine the factor group $A(G)/I(G)$, where $A(G)$ is the group of all birational and biregular automorphisms of G and $I(G)$ is the subgroup of inner automorphisms. $A(G)/I(G)$ can be identified with a subgroup of $A(\mathfrak{g})/G$. The author states, with indications of proof, that $A(G)=I(G)$ except when $\mathfrak{g}$ is of type D_4. For this exceptional case, a result of Dieudonné is used to show that $A(G)/I(G)$ is the cyclic group of order 3. *S. Helgason* (Zbl **82**, 26)

8015:

Roquette, Peter. Bericht über algebraische Gruppen. Jber. Deutsch. Math. Verein. **62**, Abt. 1, 53–84 (1959).

This is an expository article on the theory of algebraic groups. Beginning with the definition of algebraic varieties, the author explains what are algebraic groups and states some of their important properties. No new theorem is proved in this article. Furthermore, most of the facts explained in this article are not accompanied by any sketch of proof. *M. Nagata* (Kyoto)

8016:

Barsotti, Iacopo. Risultati e problemi nella teoria delle varietà gruppali. Rend. Sem. Mat. Messina **4** (1958/59), 1–48.

Malgré son titre, le présent rapport concerne surtout les variétés abéliennes, et rend compte des résultats obtenus de 1948 à 1958 par les méthodes algébrico-géométriques. Les principaux problèmes qui se posent concernent les variétés sur un corps de caractéristique non nulle; un grand nombre de résultats, qui sont presque immédiats dans le cas complexe par les méthodes transcendantes, demandent d'abord à être reformulés convenablement, et nécessitent ensuite des méthodes propres à la caractéristique non nulle. L'auteur donne ici, sans démonstrations, un aperçu sur les problèmes et les méthodes créées (par lui-même dans un certain nombre de cas) pour les résoudre.

Les problèmes se groupent en quatre rubriques: (a) Classification des extensions d'une variété abélienne par le groupe multiplicatif G_m. (b) Idem, en remplaçant G_m par le groupe additif G_a. (c) Démonstration de l'isomorphisme d'une variété abélienne avec sa biduale, et de l'identité de l'équivalence arithmétique et l'équivalence algébrique des diviseurs sur une variété abélienne. (d) Définition de matrices p-adiques représentant les homomorphismes de variétés abéliennes, lorsque p est la caractéristique du corps de base, et analogues aux matrices l-adiques de Weil ($l\neq p$).

La solution du problème (a) résulte immédiatement de celle du problème (c), une fois connue la théorie de la variété de Picard; le problème (b) se résout en caractéristique 0 au moyen des différentielles de seconde espèce, mais celles-ci deviennent inadéquates en caractéristique $p\neq 0$; il faut les remplacer par les "répartitions" de l'auteur, le premier groupe de cohomologie de J.-P. Serre, ou les diviseurs additifs du rapporteur. Le problème (c) se résout par une étude approfondie des systèmes de facteurs, des différentielles, et des répartitions. {Depuis la parution de ce rapport, le rapporteur a publié une méthode différente pour la résolution du problème (c) [cf. Ann. of Math. **71** (1960), 315–351; MR **22** #6814].} Le problème (d) nécessite l'intervention de la théorie des groupes formels commutatifs (J. Dieudonné) dont on trouvera ici un excellent résumé; à ce propos, la fin du rapport de l'auteur contient des résultats nouveaux sur le "calcul différentiel des vecteurs de Witt", qui permettent d'établir une "dualité" entre les modules p-adiques attachés à une variété abélienne et sa duale [Ann. Scuola. Norm. Sup. Pisa (3) **13** (1959), 303–372; MR **22** #4718].

Parmi les problèmes ouverts signalés à la fin, le problème 8.5 (théorème de Frobenius) a été résolu récemment par M. Nishi [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 333–350; MR **22** #6815]. *P. Cartier* (Orsay)

8017:

Koizumi, Shoji. On specialization of the Albanese and Picard varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1960), 371–382.

Let k be a field with a discrete valuation v of rank 1 whose valuation ideal is $\mathfrak{p}$. An abelian variety A defined over k is said to be without defect for $\mathfrak{p}$ if the specialization of A with respect to $\mathfrak{p}$ is an abelian variety, say $\tilde{A}$, and if the specialization of the graph of the composition law on A gives that on $\tilde{A}$. Under this terminology, the following theorems are proved:

(1) Let V be a $\mathfrak{p}$-simple variety defined over k such that V and its specialization $\tilde{V}$ with respect to $\mathfrak{p}$ are nonsingular in codimension 1. Let f be a rational mapping from V into an abelian variety A, where f and A are defined over k. If $f(V)$ generates A, then there is an abelian variety A_1 defined over k, without defect for $\mathfrak{p}$, which is k-isomorphic to A.

(2) With the same V and $\tilde{V}$ as in (1), there are models of the Albanese variety and the Picard variety of V which are without defect for $\mathfrak{p}$.

(3) Let A be an abelian variety defined over k. If the specialization of A with respect to $\mathfrak{p}$ is a nonsingular variety, then A is without defect for $\mathfrak{p}$.

(4) Let A be an abelian variety defined over k, without defect for $\mathfrak{p}$. Then: (i) There is a model $P(A)$ of the Picard variety of A which is defined over k and without defect for $\mathfrak{p}$. (ii) The specialization $\widetilde{P(A)}$ of $P(A)$ with respect to

$\mathfrak{p}$ is the Picard variety of the specialization $\tilde{A}$ of A with respect to $\mathfrak{p}$. (iii) If $\Phi : \mathfrak{g}_a(A) \to P(A)$ is a canonical mapping, then a canonical mapping $\tilde{\Phi} : \mathfrak{g}_a(\tilde{A}) \to \widetilde{P(A)}$ can be chosen such that if $(X) \in \mathfrak{g}_a(A)$ and if $\tilde{X}$ is a specialization of X with respect to $\mathfrak{p}$, then $\tilde{\Phi}(\tilde{X})$ is a uniquely determined specialization of $\Phi(X)$ over the specialization $X \to \tilde{X}$.

M. Nagata (Kyoto)

8018:

Cartier, Pierre. Sur un théorème de Snapper. Bull. Soc. Math. France **88** (1960), 333–343.

Let X be an algebraic variety, defined over an algebraically closed groundfield. The sheaf of local rings of X is denoted by O_X and the category of the algebraic, coherent sheaves over X by C.

If L is a sheaf over X which is locally isomorphic with O_X, then $L \in C$ and one defines: (1) $L^0 = O_X$; (2) if m is a positive rational integer, L^m is the tensor product of m sheaves isomorphic with L; (3) L^{-1} is the sheaf of germs of homomorphisms of L into O_X; (4) if m is a positive rational integer, L^{-m} is the tensor product of m sheaves isomorphic with L^{-1}.

The theorem mentioned in the title of the paper under review is the following (theorem 1): Assume that X is projective, that $F \in C$ and that $L_1, \cdots, L_n$ are sheaves over X which are locally isomorphic with O_X. Then, the Euler-Poincaré characteristic $\chi(X, F \otimes L_1^{m_1} \otimes \cdots \otimes L_n^{m_n})$ is a polynomial in $m_1, \cdots, m_n$ with rational numbers as coefficients, for all rational integers $m_1, \cdots, m_n$. The degree of this polynomial does not exceed the dimension of the support of F.

Theorem 1 holds without any restrictions on the projective variety X. The proof of this theorem, given by the reviewer in J. Math. Mech. **8** (1959), 967–992 [MR **22** #44], is entirely elementary and uses no theorems of the Riemann-Roch type. In the special case in which X is irreducible and nonsingular, theorem 1 follows from the Riemann-Roch-Hirzebruch-Grothendieck theorem.

The author does two things in the paper under review: he generalizes and he simplifies.

In order to understand his generalization, one must know what is meant by an "additive function on C". This is a function $\lambda : C \to G$, where G is an abelian group, and which has the property that for each exact sequence $0 \to F' \to F \to F'' \to 0$, where $F', F, F'' \in C$, $\lambda(F') - \lambda(F) + \lambda(F'') = 0$. For example, if X is complete and F runs through C, the Euler-Poincaré characteristic $\chi(X, F)$ is an additive function. The author gives an example of an additive function which is different from the Euler-Poincaré characteristic.

He then proves the following (theorem 2): Let X be either quasi-projective, or irreducible and nonsingular. Let λ be an additive function on C and let $F, L_1, \cdots, L_n$ have the same meaning as in theorem 1. Then

$$\lambda(F \otimes L_1^{m_1} \otimes \cdots \otimes L_n^{m_n})$$

is a polynomial in $m_1, \cdots, m_n$ for all rational integers $m_1, \cdots, m_n$. The degree of this polynomial does not exceed the dimension of the support of F.

It is clear that theorem 2 implies theorem 1. The author's proof of theorem 2 is again elementary, in the sense that he uses no theorems of the Riemann-Roch type. His proof is based on the following key-result.

Theorem 3 (Théorème on page 337): Let X be either quasi-projective or irreducible and nonsingular. Let r be a positive integer and λ an additive function on C, such that $\lambda(F) = 0$ for all $F \in C$ with dimension (support $(F)) < r$. Then, for all $F \in C$ with dimension (support $(F)) \leq r$ and all sheaves L which are locally isomorphic with O_X, $\lambda(F) = \lambda(F \otimes L)$.

Once theorem 3 has been proved, theorem 2 follows fairly easily. The author's arrangement, theorem 3$\Rightarrow$ theorem 2$\Rightarrow$theorem 1, constitutes a simplification and generalization of the reviewer's results.

The coefficients of the rational polynomial

$$\chi(X, F \otimes L_1^{m_1} \otimes \cdots \otimes L_n^{m_n})$$

of theorem 1 were investigated by the reviewer in J. Math. Mech. **9** (1960), 123–139 [MR **22** #5636]. The author derives the identities to which this investigation gives rise in a simpler way, by introducing a formalism analogous to the one used by Hirzebruch in *Neue topologische Methoden in der algebraishen geometrie* [Springer, Berlin, 1956; MR **18**, 509], pp. 130–136. In fact, he uses this formalism in his proof that theorem 3$\Rightarrow$theorem 2.

{The author's paper (and several other recent papers as well) poses a question. The principal theorems in his paper hold for quasi-projective varieties and for irreducible, nonsingular varieties. This indicates that there should be a "sound" class of varieties which contains the two types, just mentioned.}

E. Snapper (Bloomington, Ind.)

LINEAR ALGEBRA

See also 7974, 8394, 8421, 8422, 8423, B8616.

8019:

Smith, G. F. On the minimality of integrity bases for symmetric 3×3 matrices. Arch. Rational Mech. Anal. **5**, 382–389 (1960).

The author proves that the basis for invariants of five or fewer symmetric 3×3 matrices under the orthogonal group obtained by Rivlin and Spencer [same Arch. **4** (1960), 214–230; MR **22** #715] is in fact a minimal basis.

J. A. Todd (Cambridge, England)

8020:

Kuiper, N. H. Barycentric calculus and the beginning of vectors. Euclides (Groningen) **35** (1959/60), 113–126. (Dutch)

Expository.

8021:

Janekoski, V. Sur la formule du double produit vectoriel et la propriété de distributivité de la multiplication vectorielle. Bull. Soc. Math. Phys. Macédoine **10** (1959), 39–42. (Serbo-Croatian. French summary)

Author's summary: "Dans cette Note l'auteur expose quelques procédés vectoriels pour démontrer la formule du double produit vectoriel $(a \times b) \times c = (c \cdot a)b - (b \cdot c)a$ et donne une transformation par laquelle, à l'aide de la formule du double produit vectoriel, on peut démontrer la propriété de distributivité de la multiplication vectorielle par rapport à l'addition."

8022:

Marcus, Marvin; Minc, Henryk. The maximum number of equal non-zero subdeterminants. Arch. Math. **11** (1960), 95–100.

The authors consider the two equivalent problems: (i) determine the largest integer $p_{k,n}$ for which there exists a $k\times n$ matrix with $p_{k,n}$ equal non-zero k-square subdeterminants; (ii) determine the largest integer $p_{k,n}$ for which a pure vector $a_1\wedge\cdots\wedge a_k$ in the kth Grassmann product space has $p_{k,n}$ equal non-zero coordinates. It is proved that $p_{k,n}<\binom{n}{k}$ unless $k=n-1$ or n, and that, for $n>3$, $p_{2,n}=[n^2/3]$, where [ ] denotes the largest integer function. A corollary to the last result states than an n-square skew-symmetric matrix of rank 2 has at most $[n^2/3]$ equal non-zero entries above the main diagonal.

K. Goldberg (Washington, D.C.)

8023:

Nakamura, Yoshio. On Witt ring of quadratic forms. J. Math. Soc. Japan **12** (1960), 187–191.

Let k be a commutative field of characteristic $\neq 2$. If R is a vector space over k with a regular symmetric bilinear form, R permits direct decompositions $R=R_0\oplus N_1\oplus N_2\oplus\cdots$ with the following properties: R_0 and the N_i are mutually orthogonal; the form is definite in R_0; each N_i is a two-space spanned by two isotropic vectors with the scalar product 1. The "kernel" R_0 of R is uniquely determined up to isomorphisms. A "type" over k is the set of all the vector spaces over k whose kernels are isomorphic to the same R_0. Define type R+type S=type $(R\oplus S)$; type $R\cdot$type S=type of the Kronecker (=tensor) product of R and S. The types over k then form a commutative ring, the Witt ring over k; cf. E. Witt, J. Reine Angew. Math. **176** (1936), 31–44. The author determines the structure of this ring in the following cases: (i) k is finite; (ii) k is complete with respect to a discrete non-archimedian valuation whose residue class field is finite and has a characteristic $\neq 2$; (iii) k is algebraic over the rational field.

P. Scherk (Toronto)

8024:

Fan, Ky. Note on M-matrices. Quart. J. Math. Oxford Ser. (2) **11** (1960), 43–49.

The nth order square matrix A equal to $\rho I-A_1$, where A_1 has non-negative elements and ρ is greater than the absolute value of every characteristic root of A_1, is called an M-matrix. If α denotes a subset of the set $\{1, 2, \cdots, n\}$, let $A(\alpha)$ denote the principal minor of A formed by the rows and columns with indices in α, and, if $\emptyset$ is the empty set, set $A(\emptyset)=1$. The main result is that if A, B are M-matrices of order n such that $a_{ij}\leq b_{ij}$ for all i, j, then $\Phi(A;\alpha,\beta,\gamma)\leq\Phi(B;\alpha,\beta,\gamma)$, where α, β, γ are subsets of $\{1, 2, \cdots, n\}$ and

$$\Phi(A;\alpha,\beta,\gamma)=\frac{A(\alpha\cap\beta)A(\alpha\cap\gamma)A(\beta\cap\gamma)A(\alpha\cup\beta\cup\gamma)}{A(\alpha)A(\beta)A(\gamma)A(\alpha\cap\beta\cap\gamma)}$$

H. S. A. Potter (Aberdeen)

8025:

Marcus, M.; Ree, R. Diagonals of doubly stochastic matrices. Quart. J. Math. Oxford Ser. (2) **10** (1959), 296–302.

The permanent of an $n\times n$ matrix $A=(a_{ij})$ is defined as

$$\sum_\sigma a_{1,\sigma(1)}\cdots a_{n,\sigma(n)},$$

where the summation extends over all permutations σ of $1, \cdots, n$. It was conjectured 35 years ago by van der Waerden that, as S varies over the set of doubly-stochastic (d.s.) $n\times n$ matrices, the permanent of S attains its minimum value uniquely for the matrix all of whose elements are equal to n^{-1}. The truth of this conjecture remains as yet unproved, though progress was made recently by M. Marcus and M. Newman [Duke Math. J. **26** (1959), 61–72; MR **21** #3432]. It was pointed out by P. Erdös to the authors of the present paper that the truth of van der Waerden's conjecture would imply the following result, referred to as (b). If $S=(s_{ij})$ is a d.s. $n\times n$ matrix, then there exists a permutation σ such that $s_{i,\sigma(i)}>0$ $(1\leq i\leq n)$ and

$$\sum_{i=1}^n s_{i,\sigma(i)}\geq 1.$$

Making use of Birkhoff's theorem [Univ. Nac. Tucumán. Rev. Ser. A **5** (1946), 147–151; MR **8**, 561], the authors establish the following strengthened version of this result. Let S be a d.s. $n\times n$ matrix, and write $R=(r_{ij})=S'S$. If $g(t_1, \cdots, t_n)$ is a real-valued function defined for $0\leq t_j<1$ $(1\leq j\leq n)$ and convex with respect to each variable, then there exists a permutation σ such that $s_{i,\sigma(i)}>0$ $(1\leq i\leq n)$ and

$$g(s_{1,\sigma(1)}, \cdots, s_{n,\sigma(n)})\geq g(r_{11}, \cdots, r_{nn}).$$

The special case $g(t_1, \cdots, t_n)=t_1+\cdots+t_n$ leads easily to (b).

Several other results of this type are proved, and we mention two of them. (i) If the d.s. $n\times n$ matrix S has k characteristic roots of modulus 1, then there exists a permutation σ such that $s_{i,\sigma(i)}>0$ $(1\leq i\leq n)$ and

$$\sum_{k=1}^n s_{i,\sigma(i)}\geq k.$$

Since 1 is a characteristic root of every d.s. matrix, this is again seen to include (b). (ii) If S is a d.s. $n\times n$ matrix, then there exists a permutation σ such that, for $1\leq i\leq n$,

$$s_{i,\sigma(i)}\geq\frac{1}{k(k+1)}\quad(n=2k)$$

$$\geq\frac{1}{(k+1)^2}\quad(n=2k+1).$$

The proof of this result depends on a well-known combinatorial theorem of Frobenius and König.

{Reviewer's remark: The condition that $g(t_1, \cdots, t_n)$ is convex with respect to each variable is slightly misleading. However, it is clear from the context that $g(t_1, \cdots, t_n)$ is assumed to be a convex function of the vector $(t_1, \cdots, t_n)$.}

L. Mirsky (Sheffield)

8026:

Mirsky, L. An algorithm relating to symmetric matrices. Monatsh. Math. **64** (1960), 35–38.

The relations between the diagonal and the characteristic roots of a real symmetric matrix have been completely analysed previously [see, e.g., Mirsky, J. London Math. Soc. **33** (1958), 14–21; MR **19**, 1034]. Here rational necessary and sufficient conditions are given for the exist-

ence of a real symmetric matrix with prescribed diagonal and characteristic polynomial.

O. Taussky-Todd (Pasadena, Calif.)

8027:

Brauer, Alfred. Stochastic matrices with a non-trivial greatest positive root. Duke Math. J. **27** (1960), 291–295.

The author proves the following theorem: Let A be a generalized stochastic matrix with the property that the off-diagonal elements of one particular row are less than or equal to the other elements in the same column. Then A has a positive non-trivial characteristic root η whose modulus is not less than the moduli of all the other characteristic roots of A, unless the non-trivial roots of A all vanish.

The paper also contains inequalities for η and an analogous theorem on doubly stochastic matrices.

W. Ledermann (Manchester)

8028:

Ostrowski, Alexander; Schneider, Hans. Bounds for the maximal characteristic root of a non-negative irreducible matrix. Duke Math. J. **27** (1960), 547–553.

Let $A=(a_{\nu\mu})$ be an $n\times n$ non-negative irreducible matrix with row sums $r_\nu=\sum_\mu a_{\nu\mu}$ $(\nu,\mu=1,2,\cdots,n)$. Put $R=\max_\nu r_\nu$, $r=\min_\nu r_\nu$. It is well known that, if $r<R$, the maximal root, ω, of A satisfies the relations $r<\omega<R$. Several authors have recently improved this result by determining quantities L, U, which can easily be calculated from the elements $a_{\nu\mu}$, such that $r<L\leq\omega\leq U<R$. The present authors show that this chain of inequalities holds when $L=r+\varepsilon(\rho-r)$, $U=R-\varepsilon(R-\rho)$, where

$$\rho=(1/n)\sum_\nu r_\nu,\quad \varepsilon=(\kappa/(R-\lambda))^{n-1},\quad \lambda=\min_\nu a_{\nu\nu},$$
$$\kappa=\min(a_{\nu\mu}|_{\nu\neq\mu},\ a_{\nu\mu}>0).$$

By more elaborate arguments this result is further improved, but the calculations of the revised bounds L, U are more complicated.

W. Ledermann (Manchester)

8029:

Lancaster, P. Inversion of lambda-matrices and application to the theory of linear vibrations. Arch. Rational Mech. Anal. **6** (1960), 105–114.

Let $D(\lambda)\equiv A_0\lambda^l+A_1\lambda^{l-1}+\cdots+A_{l-1}\lambda+A_l$, where $A_0,\cdots,A_l$ are $n\times n$ matrices with real elements, A_0 is non-singular and λ is scalar. Let $\lambda_1,\cdots,\lambda_{ln}$ be the roots of $\det D(\lambda)=0$, and denote by Λ the diagonal matrix with these roots as diagonal elements. The $n\times ln$ matrices Q, R are constructed from column vectors q_i, r_i such that $D(\lambda_i)q_i=0$, $[D(\lambda_i)]'r_i=0$. The main result proved is that if $D(\lambda_i)$ has degeneracy (i.e., nullity) equal to the multiplicity of λ_i for $i=1,\cdots,ln$, then the vectors q_i, r_i can be so normalised that

$$\lambda^r[D(\lambda)]^{-1}=Q\Lambda^r(I\lambda-\Lambda)^{-1}R'\quad\text{for } r=0,1,\cdots,l-1$$

and

$$\lambda^l[D(\lambda)]^{-1}=Q\Lambda^l(I\lambda-\Lambda)^{-1}R'+A_0^{-1}\quad\text{if }\lambda\neq\lambda_i.$$

H. S. A. Potter (Aberdeen)

ASSOCIATIVE RINGS AND ALGEBRAS

8030:

Radu, N. Le lemme de Krull et la structure des anneaux. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 183–191. (Romanian. Russian and French summaries)

Author's summary: "Généralisant les anneaux de Dedekind, Keizo Asano [J. Math. Soc. Japan **3** (1951), 82–90; MR **13**, 313] a introduit, dans les anneaux commutatifs $\mathcal{O}$ à élément-unité, les hypothèses: I. La condition maximale pour les idéaux. II. L'irréductibilité de tout intervalle $P>P^2$, où P est un idéal maximal.

"L'auteur remplace la première de ces hypothèses par deux de ses conséquences: (1) Le lemme de Krull: tout idéal véritablement $A<\mathcal{O}$, divise un produit de diviseurs par $A\supset P_1P_2\cdots P_n$, $P_i\supset A$ $(i=1,\cdots,n)$. (2) La condition maximale pour les idéaux premiers. Il considère également un autre axiome (Grell): (3) Les idéaux non nuls ne peuvent être des intersections irréductibles d'une infinité d'idéaux.

"C'est ainsi que l'on démontre, lorsqu'il s'agit d'hypothèses plus faibles, (1) + (2) + II, le théorème d'Akizuki relatif à la décomposition des idéaux en produits d'idéaux premiers. Un autre résultat important montre l'existence d'une base finie pour les idéaux maximaux dans les conditions (1) + (2) + (3). La méthode utilisée est fondée sur la notion d'idéal successeur."

B. Harris (Princeton, N.J.)

8031:

Zemmer, J. L. A Boolean geometry for the integers. Amer. Math. Monthly **67** (1960), 56–57.

The author defines a mapping $n\to\varphi(n)$ of the ring of integers J into the boolean algebra $\mathfrak{B}$ of all infinite sequences of the elements 0, 1 of a boolean algebra of two elements (boolean operations in $\mathfrak{B}$ defined componentwise) by associating with $n\in J$ the element $\{a_i\}$ of $\mathfrak{B}$, where $a_i=0$ or 1 according as the ith prime p_i does or does not divide n, respectively. The mapping φ has properties (1) $\varphi(n)=\varphi(0)$ if and only if $n=0$, (2) $\varphi(n+m)\subset\varphi(n)\cup\varphi(m)$, (3) $\varphi(m\cdot n)=\varphi(m)\cap\varphi(n)$, (4) $\varphi(n)=\varphi(1)$ if and only if $n=\pm1$. An arbitrary ring R is called boolean-valued provided there exists a mapping ψ of R into a boolean algebra $\mathfrak{A}$, where ψ has properties (1), (2), (3). The mapping ψ is called a valuation and $\mathfrak{A}$ is a valuation algebra of R. A boolean-valued metric is defined for such rings R by defining $\psi(y-x)$ as the distance $\rho(x,y)$ of $x,y\in R$. Two theorems are proved: (I) If $\mathfrak{B}$ is a valuation algebra of R and the two rings R, J are congruent (with respect to $\mathfrak{B}$), then the rings are isomorphic; and (II) the only motions of J are translations and reflections.

L. M. Blumenthal (Columbia, Mo.)

8032:

Utumi, Yuzo. On continuous regular rings and semisimple self injective rings. Canad. J. Math. **12** (1960), 597–605.

The regular ring S is said to be l-continuous if the lattice L of principal l-ideals of S is a complete, complemented, modular lattice having the following property: For each subset T of L and each $b\in L$, if $(\bigcup_i t_i)\cap b=0$ for each finite subset $\{t_1,\cdots,t_n\}$ of T, then $t\cap b=0$, where t is the union of all elements of T. In what follows, let Q designate

the maximal left quotient ring of S and $E(Q)$ designate the set of all idempotent elements of Q. If S is l-continuous, then every closed l-ideal A (i.e., no l-ideal $A' \supset A$ is an essential extension of A) of S is principal. From this, it is proved that $E(Q) \subset S$ if S is l-continuous. Next, it is proved that if Q is any semisimple left self-injective ring for which every nonzero ideal A has index $i(A) \neq 1$ (i.e., A contains nonzero nilpotent elements), then Q is generated by $E(Q)$ and also by its set of nonsingular elements. From this it follows that if S is a left continuous ring having no nonzero ideals of index 1, then S is self-injective (i.e., $S=Q$). An example is given to show that the assumption about the index of the ideals of S cannot be dropped. If S is a regular ring with unity such that $i(S)=1$, then the following statements are shown to be equivalent: (i) S is continuous; (ii) the Boolean algebra $E(S)$ is complete; (iii) the Boolean ring $E(S)$ is self-injective. This result is related to the work of Brainerd and Lambek [Canad. Math. Bull. **2** (1959), 25–29; MR **21** #12].

R. E. Johnson (Rochester, N.Y.)

8033:

Schenkman, Eugene; Scott, W. R. A generalization of the Cartan-Brauer-Hua theorem. Proc. Amer. Math. Soc. **11** (1960), 396–398.

The result of this paper ("a noncommutative division ring contains no non-trivial subinvariant division subring") in a slightly restricted form was announced by Schenkman in same Proc. **9** (1958), 231–235 [MR **21** #3443]; the gap (discovered by one of the present authors) in the proof given there is closed here.

Carl Faith (Princeton, N.J.)

8034:

Auslander, Maurice; Goldman, Oscar. Maximal orders. Trans. Amer. Math. Soc. **97** (1960), 1–24.

Let R be an integrally closed noetherian domain with quotient field K, and let Σ be a fixed central simple algebra over K. An order over R in Σ is a subring of Σ which is a finitely generated R-module and spans Σ over K; Λ is maximal if it is not properly contained in any other order. Let $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$, $\Lambda^{**} = \mathrm{Hom}_R(\Lambda^*, R)$. Then Λ^{**} can be identified with an R-submodule of Σ containing Λ. The authors prove that Λ is a maximal order if and only if $\Lambda = \Lambda^{**}$ and $\Lambda \otimes_R R_P$ is a maximal order over R_P for every minimal prime ideal P in R.

Now let $\mathfrak{O}$ be a discrete rank one valuation ring with quotient field K, and Σ a central simple K-algebra. An order Λ over $\mathfrak{O}$ is maximal if and only if the radical N of Λ is a maximal two-sided ideal in Λ, and Λ is hereditary (i.e., every submodule of a projective Λ-module is projective). Using this result, the authors prove the following theorem. Let R be a Dedekind ring with quotient field K, and Σ a central simple K-algebra. An order Λ over R is maximal if and only if Λ is hereditary and for any non-zero prime ideal p in R, there is one and only one maximal two-sided ideal P in Λ containing p.

Returning to the case of an order Λ over a discrete rank one valuation ring $\mathfrak{O}$, the authors prove that if Λ is maximal then Λ is a principal ideal ring. Any two maximal orders over $\mathfrak{O}$ in Σ are conjugate by an inner automorphism of Σ. A maximal order Λ over $\mathfrak{O}$ is a full matrix algebra over a maximal $\mathfrak{O}$-order in a division algebra.

It is a known result that maximal orders in a full matrix algebra over the quotient field of a Dedekind domain R are endomorphism rings of finitely generated projective modules over the Dedekind ring. The authors prove the following generalization. A domain R is called a regular domain if R is noetherian and the ring of quotients R_P is a regular local ring for every prime ideal P of R. Let R be a regular domain with quotient field K, V a finite dimensional K-space, and $\Sigma = \mathrm{Hom}_K(V, V)$. Let Λ be an order over R in Σ which is R-projective. Then Λ is maximal if and only if $\Lambda = \mathrm{Hom}_R(E, E)$, where E is a finitely generated projective R-submodule of V.

Several of the results concerning orders over discrete rank one valuation rings were proved under the additional hypothesis of completeness by Deuring [*Algebren*, J. Springer, Berlin, 1935].

C. W. Curtis (Madison, Wis.)

8035:

Aeberli, G. Der Zusammenhang zwischen quaternären quadratischen Formen und Idealen in Quaternionenringen. Comment. Math. Helv. **33** (1959), 212–239.

Let $\mathfrak{A}$ be an algebra of quaternions over the field of rational numbers. As usual in the arithmetical theory of algebras, the author considers those ideals $\mathfrak{a}$ of $\mathfrak{A}$ whose right and left orders are maximal. Two such ideals $\mathfrak{a}$, $\mathfrak{b}$ are called equivalent (in the narrow sense), if $\mathfrak{a} = \rho\mathfrak{b}\sigma$ with $\rho, \sigma \in \mathfrak{A}$ and $\mathrm{Norm}(\rho\sigma) > 0$. There are finitely many corresponding ideals classes. They form under multiplication an algebraic structure which Brandt has called "groupoid". The purpose of this paper is to show that this groupoid is isomorphic to another groupoid consisting of classes of quaternary quadratic forms. Let $\alpha = (\alpha_0\alpha_1\alpha_2\alpha_3)$ be a basis of the ideal $\mathfrak{a}$ over the rational integers. Put $\mathrm{Norm}(\sum_{i=0}^{3} \alpha_i x_i) = \mathrm{Norm}(\mathfrak{a}) \cdot F(x_0, x_1, x_2, x_3)$, where the norm of the ideal $\mathfrak{a}$ is taken to be a positive integer. Then F is the quadratic form corresponding to the ideal $\mathfrak{a}$. If certain conventions about the arrangement of the basis elements α_i are made (such that the determinant of the substitution from one basis into another is positive) then it can be shown that F is uniquely determined up to equivalence by the ideal class of $\mathfrak{a}$. Here two forms are said to be equivalent (in the narrow sense) if one can be transformed into the other by a linear substitution with integral coefficients and determinant $+1$. The correspondence $\mathfrak{a} \rightarrow F$ then gives the above-mentioned isomorphism of the groupoid of ideal classes of $\mathfrak{A}$ onto a groupoid of form classes. The multiplication of ideal classes corresponds to the so-called "composition" of forms, defined by bilinear transformations of the variables. The forms F corresponding to ideals $\mathfrak{a}$ of $\mathfrak{A}$ have the following properties: (1) The coefficients of F are rational integers with greatest common divisor 1 (i.e., F is integral and primitive); (2) F is a stem form (Stammform) in the following sense. The discriminant of F has minimal absolute value among the discriminants of those integral primitive forms G which can be transformed into a multiple of F by a linear substitution of the variables with rational coefficients. It is shown that all the forms F corresponding to ideals $\mathfrak{a}$ of $\mathfrak{A}$ have the same discriminant D, which is a square. Every form F satisfying (1) and (2) and having the discriminant D corresponds to an ideal of $\mathfrak{A}$. The result of this paper extends a similar theorem concerning the ideal classes of quadratic fields (instead of quaternion algebras) and their corresponding binary norm

forms. [Literature: Deuring, *Algebren*, J. Springer, Berlin, 1935; and the papers of Brandt cited there.]
P. Roquette (Tübingen)

8036:
Micali, Artibano. Algèbre symétrique d'un idéal. C. R. Acad. Sci. Paris **251** (1960), 1954–1956.

Let $\mathfrak{A}$ ($\neq 0$) be an ideal of a commutative ring A, let $S(\mathfrak{A})$ be the symmetric algebra of the A-module $\mathfrak{A}$, and let $R(\mathfrak{A})=\sum_{n\geq 0}\mathfrak{A}^nT^n$ be the Rees ring associated with $\mathfrak{A}$. The homomorphism θ of $\mathfrak{A}$ into $R(\mathfrak{A})$ defined by $\theta(a)=aT$ for all $a\in\mathfrak{A}$ can be extended in a unique way to an epimorphism θ of $S(\mathfrak{A})$ onto $R(\mathfrak{A})$. The main result asserts that if $\mathfrak{A}$ is generated by an A-sequence, then θ is an isomorphism. Further results relating θ to mappings of polynomial rings into $S(\mathfrak{A})$ and $R(\mathfrak{A})$, giving information about A-sequences and identifying Ker θ with the torsion submodule of $S(\mathfrak{A})$, are also announced.
H. T. Muhly (Iowa City, Iowa)

8037:
Nöbauer, Wilfried. Zur Theorie der Vollideale. II. Monatsh. Math. **64** (1960), 335–348.

The present paper carries further the discussion of full ideals given by the author in three earlier papers [Math. Ann. **134** (1958), 248–259; J. Reine Angew. Math. **201** (1959), 207–220; Monatsh. Math. **64** (1960), 176–183; MR **20** #4549; **21** #7225; **22** #5652]. The paper is concerned with the following three problems. (1) Reduction of the study of full ideals of a polynomial ring in k indeterminates to the case of $\nu<k$ indeterminates. This leads to the generalization of the notion of enveloping ideal. (2) Determination of when sum and product of Restpolynomideale are again Restpolynomideale. (3) The derivative of a full ideal was defined in the third paper cited above. A D-full ideal is defined as a full ideal that coincides with its derivative. Some properties of D-full ideals are obtained. In particular the D-full ideals of the ring $F[x]$, where F is a finite field, are determined. *L. Carlitz* (Durham, N.C.)

8038:
Jaffard, Paul. ★Théorie de la dimension dans les anneaux de polynomes. Mémor. Sci. Math., Fasc. 146. Gauthier-Villars, Paris, 1960. 79 pp. 15 NF; \$3.25.

Étant donné un anneau commutatif A, la dimension $\dim(A)$ de A est la borne supérieure des longueurs des chaines d'idéaux premiers de A. On note $A^{(n)}$ l'anneau des polynomes à n variables sur A. Si A est intègre, la dimension valuative $\dim_v(A)$ est la borne supérieure des longueurs des chaines d'anneaux de valuation du corps des fractions de A qui contiennent A; dans le cas général $\dim_v(A)$ est la borne supérieure des $\dim_v(A/\mathfrak{p})$ pour $\mathfrak{p}$ premier.

Le chapitre I de ce livre contient des préliminaires sur les points suivants: spécialisations, valuations, inclusion des anneaux de valuations, prolongement des valuations aux extensions transcendantes; idéaux premiers des anneaux de polynômes.

Etant donnés deux idéaux premiers $\mathfrak{p}$, $\mathfrak{q}$ d'un anneau A tels que $\mathfrak{q}\subset\mathfrak{p}$, on note $\delta(\mathfrak{q},\mathfrak{p})$ la borne supérieure des longueurs des chaines de valuations de $A/\mathfrak{q}$ ayant $\mathfrak{p}/\mathfrak{q}$ pour centre sur $A/\mathfrak{q}$. On dit que A est sans-poids (resp. lèger) si $\delta(\mathfrak{q},\mathfrak{p})=0$ quels que soient $\mathfrak{q}$ et $\mathfrak{p}$ (resp. chaque fois que $\mathfrak{p}$ est un surideal premier immédiat de $\mathfrak{q}$). Le chapitre II étudie ces notions. Comportment par localisation et par extension entière. Un anneau de Prüfer est sans-poids. Le résultat fondamental est que la borne supérieure des longueurs des chaines $(\mathfrak{P}_i)_{0\leq i\leq s}$ d'idéaux premiers de $A^{(n)}$ vérifiant $\mathfrak{P}_i\cap A=\mathfrak{q}$ pour $i\neq s$, et $\mathfrak{P}_s=\mathfrak{p}A^{(n)}$ est égale à $1+\inf(n,\delta(\mathfrak{q},\mathfrak{p}))$. Il existe une chaine $(\mathfrak{Q}_j)$ d'idéaux premiers de $A^{(n)}$ contenant les idéaux $(\mathfrak{Q}_j\cap A)A^{(n)}$ et de longueur $\dim(A^{(n)})$. On en déduit que $\dim(A^{(n)})=n+\dim(A)$ lorsque A est sans-poids.

Dans le cas général, le chapitre III montre qu'il existe deux entiers π et δ tels que $\dim(A^{(n)})=(1+\pi)n+\delta$ pour n grand ($0\leq\pi\leq\delta$ et $\pi\leq\dim(A)$). Étude du cas où $\dim(A)=1$. Si $A^{(n)}$ est lèger pour tout n, on a $\dim(A^{(n)})=n+\dim(A)$; comme un anneau noethérien est lèger, on retrouve le résultat de Krull que $\dim(A^{(n)})=n+\dim(A)$ pour A noethérien.

Le chapitre IV traite de la dimension valuative $\dim_v(A)$. On a $\dim(A)\leq\dim_v(A)$, avec égalité si A est noethérien ou est un anneau de Prüfer; l'égalité se transmet de A à $A^{(n)}$, et, si A est équidimensionnel, de A à ses anneaux quotients et anneaux de fractions. Pour que $\dim_v(A)$ soit finie, il faut et il suffit que $\pi=0$; on a alors $\dim(A^{(n)})=n+\dim_v(A)$ pour n grand. Construction d'anneaux dans lesquels les entiers π et δ prennent des valeurs finies arbitraires tells que $\pi\leq\delta$. On a $\dim_v(A^{(n)})=n+\dim_v(A)$.

Il est dommage que ce petit livre, qui est intéressant et bien rédigé, ne comporte pas d'index terminologique.
P. Samuel (Clermont-Ferrand)

8039:
Chadeyras, Marcel. Sur les anneaux semi-principaux ou de Bezout. C. R. Acad. Sci. Paris **251** (1960), 2116–2117.

A commutative integral domain A in which every finitely generated ideal is principal is called semi-principal.

In this note, it is shown that the following properties are equivalent: (a) A is a semi-local Prüfer ring; (b) A is semi-local and semi-principal; (c) A is intersection of a finite number of valuation rings in a field.

To prove the implication (a)→(b) is essential. For this purpose the author uses the proposition: Let A be a semi-local (not necessarily noetherian) domain, and P a finitely generated projective A-module; then P is free.
K. Kondô (Tokyo)

8040:
Rutsch, Martin. Remark on a note by Mr. M. Nagata on coefficient fields of complete local rings. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 295–296.

The following generalization (that includes the unequal characteristic case) of a theorem of Nagata is proved. Let R and R' be complete local rings (which may not be noetherian) with $R\subseteq R'$. Assume that the residue class fields of R and R' are of characteristic $p\neq 0$ and that $R'^p\subseteq R$. Then there is a coefficient ring of R which is extendable to a coefficient ring of R'.
H. T. Muhly (Iowa City, Iowa)

8041:
Geddes, A. On coefficient fields. Proc. Glasgow Math. Assoc. 4, 42–48 (1958).

Soient Q un anneau local (non nécéssairement noethérien), $\mathfrak{m}$ son idéal maximal, et $f:Q\to Q/\mathfrak{m}$. On suppose Q séparé et complet pour la topologie $\mathfrak{m}$-adique. Si K est un sous-corps de Q, et si F_1 est une extension séparable (algébrique ou non) de $f(K)$ contenue dans $f(Q)$, il existe

un sous corps F de Q contenant K tel que $f(F)=F_1$. Prenant pour K le sous-corps premier de Q (dans le cas d'égales caractéristiques), on retrouve l'existence de corps de représentants. Méthode de limite projective, comme dans Geddes, J. London Math. Soc. **29** (1954), 334–341 [MR **16**, 213], et Proc. London Math. Soc. (3) **6** (1956), 343–354 [MR **19**, 116]. *P. Samuel* (Clermont-Ferrand)

8042:

Narita, Masao. On the unique factorization theorem in regular local rings. Proc. Japan Acad. **35** (1959), 329–331.

M. Nagata [Amer. J. Math. **80** (1958), 382–420; MR **20** #862] has shown by homological algebraic methods that if every regular local ring of dimension 3 is a unique factorization ring, then so is every regular local ring. In this paper the author gives another proof of this theorem using only ideal-theoretic methods. The theorem proved was used by M. Auslander and Buchsbaum [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 733–734; MR **21** #2669] to prove that every regular local ring is a unique factorization ring. *F. P. Peterson* (Oxford)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

8043:

Hughes, D. R.; Kleinfeld, Erwin. Seminuclear extensions of Galois fields. Amer. J. Math. **82** (1960), 389–392.

Let R be a non-associative ring. The authors call the set of all a in R with $(ax)y=a(xy)$ [$(xa)y=x(ay)$; $(xy)a=x(ya)$] for all x, y in R, the left [middle; right] nucleus. The nucleus of R is the intersection of these three nuclei. The authors prove (theorem 1): Let R be a non-associative division ring which is a quadratic extension of a Galois field F, and suppose F is contained in the right and middle nuclei of R. Then R must be isomorphic to a ring S constructed as follows: Let S be a vector space of dimension 2 over F, having basis 1, λ and multiplication defined by $(x+\lambda y)(u+\lambda v)=(xu+\delta_0 y^\sigma v)+\lambda(yu+x^\sigma v+\delta_1 y^\sigma v)$, where σ is an arbitrary non-identity automorphism of F and δ_0, δ_1 in F are subject only to the condition that $w^{1+\sigma}+\delta_1 w-\delta_0=0$ has no solution for w in F. Conversely, given F, σ, δ_0, δ_1, satisfying the above conditions, then S will satisfy the conditions on R (theorem 2): Let R be a non-associative division ring which is a quadratic extension of a Galois field F, and suppose F is contained in the nucleus of R. Then R must be isomorphic to one of the rings S of theorem 1 with the additional stipulation that $\sigma^2=I$ and $\delta_1=0$. Conversely, all such S satisfy the conditions on R. Finally, it is shown that in order for R as in theorem 1 to exist, F must equal $\mathrm{GF}(p^n)$, $n>1$, while an R as in theorem 2 will exist for every $F=\mathrm{GF}(p^{2k})$, $k\geq 1$. *A. Rosenberg* (Berkeley, Calif.)

8044:

Schafer, R. D. Nodal noncommutative Jordan algebras and simple Lie algebras of characteristic p. Trans. Amer. Math. Soc. **94** (1960), 310–326.

A class K of simple nodal noncommutative Jordan algebras of characteristic $p\neq 2$ [see Kokoris, Canad. J. Math. **12** (1960), 488–492; MR **22** #6834] is described by beginning with the truncated polynomial ring $B_n=F(x_1, \cdots, x_n)$, $x_i^p=0$, with differentiation operators $\partial/\partial x_i$. Any algebra A of dimension p^n in K is the same vector space as B_n, but with multiplication defined by

$$fg = f\cdot g+\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij},\quad c_{ij} = -c_{ji},$$

where $f\cdot g$ is the product in B_n and c_{ij} are arbitrary elements in B_n except that at least one of them has an inverse. Using the facts that every derivation of A is a derivation of $A^+=B_n$ and that the derivations of A^+ were found by Jacobson [Duke Math. J. **10** (1943), 107–121; MR **4**, 187] to be the mappings $D:f\to\sum_{k=1}^{n}\partial f/\partial x_k\cdot a_k$ for arbitrary a_k in B_n, it is shown that D is a derivation of A if and only if

$$\sum_{k=1}^{n}\left(\frac{\partial c_{ij}}{\partial x_k}\cdot a_k+\frac{\partial a_i}{\partial x_k}\cdot c_{jk}+\frac{\partial a_j}{\partial x_k}\cdot c_{ki}\right) = 0,$$

for $1\leq i, j\leq n$. Two cases are now considered. The first is where the c_{ij} are in $F1$; that is, $c_{ij}=\varphi_{ij}1$, $\varphi_{ij}=-\varphi_{ji}$ in F, where $x_ix_j=x_i\cdot x_j+\varphi_{ij}1$. This implies a unique skew-symmetric bilinear form φ defined on $N/N\cdot N$. A lengthy proof shows that if φ has rank $2r$, the derivation algebra $D(A)$ of A has dimension $p^n-p^{n-2r}+np^{n-2r}$. Two characterizations of the (p^r-2)-dimensional Lie algebras V_r of Albert and Frank [Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954/1955), 117–139; MR **18**, 52] are given in terms of p^{2r}-dimensional nodal algebras of this type. The second case is where $n=2$. Here it is possible to choose x_1 and x_2 so that $c_{12}=c=1+\alpha x_1^{p-1}\cdot x_2^{p-1}$, and it is proved that $D(A)=C\cdot M_2$, where M_2 is a (p^2+1)-dimensional algebra consisting of certain derivations on B_2. In the last section the simple Lie algebras $L(G, d, f)$ of Block [same Trans. **89** (1958), 421–449; MR **20** #6446] are described in terms of certain algebras in K.

L. A. Kokoris (Chicago, Ill.)

HOMOLOGICAL ALGEBRA

See also 8034, 8498, 8499.

8045:

Картан, А. [Cartan, Henri]; Эйленберг, С. [Eilenberg, Samuel]. ★Гомологическая алгебра [Homological algebra]. Translated from the English by E. G. Šul'geĭfer; edited by M. M. Postnikov. Izdat. Inostr. Lit., Moscow, 1960. 510 pp. 22 r.

The original [Princeton Univ. Press, Princeton, N.J., 1956] was reviewed in MR **17**, 1040. This translation contains a brief foreword by the editor and many, usually brief, annotations by both editor and translator. The appendix by D. A. Buchsbaum, which appeared in the American edition, has been replaced by a translation of Buchsbaum's more extended paper Trans. Amer. Math. Soc. **80** (1955), 1–34 [MR **17**, 579].

8046:

Nakayama, Tadasi. A remark on relative homology and cohomology groups of a group. Nagoya Math. J. **16** (1960), 1–9.

Soient G un groupe, H un sous-groupe d'indice fini; pour tout G-module M, on considère les groupes de cohomologie relatifs $H^n(G, H; M)$ à la Hochschild-Tate (n

variant de $-\infty$ à $+\infty$), et aussi les groupes d'homologie relatifs $H_n(G, H; M)$. Lorsque H n'est pas réduit à l'élément neutre, il n'y a pas de relations entre les H^n et les H_n. On a cependant le résultat suivant:

Soit K_0 l'intersection des conjugués de H dans G; soient M et N deux G-modules, et $\kappa_0: M \to N$ un G-homomorphisme qui induise un isomorphisme $M_{K_0} \approx N^{K_0}$ (où M_{K_0} désigne le quotient de M par le sous-groupe engendré par les $u - hu$, $u \in M$, $h \in K_0$; et N^{K_0} désigne le sous-groupe des éléments de N invariants par K_0). Supposons de plus que, pour tout sous-groupe K, intersection de certains conjugués de H, on ait

$$H^0(K/K_0; M_{K_0}) = 0 = H^{-1}(K/K_0; M_{K_0}).$$

Alors κ_0 induit un isomorphisme

$$H_n(G, H; M) \approx H^{-n-1}(G, H; N)$$

pour tout entier n, positif, négatif ou nul.

H. Cartan (Paris)

GROUPS AND GENERALIZATIONS

See also 8394, 8424, B8713, B9314.

8047:

Holmes, C. V. Commutator groups of monomial groups. Pacific J. Math. **10** (1960), 1313–1318.

The author proves that every element of the infinite complete monomial group is the product of at most two commutators, and that the group is its own commutator group; the derived series is determined for certain classes of subgroups.

O. Ore (New Haven, Conn.)

8048:

Uesugi, Toshitane. On the isomorphism and the homomorphism of the bases of the pseudogroups of transformations. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 34–44.

The notions of isomorphism and homomorphism of pseudo-groups of transformations of differentiable manifolds formulated by Y. Matsushima are generalized in this paper to pseudo-groups of transformations of sets and of topological spaces.

M. Kuranishi (Princeton, N.J.)

8049:

Borůvka, Otakar. ★Grundlagen der Gruppoid- und Gruppentheorie. Hochschulbücher für Mathematik, Bd. 46. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. xii+198 pp. DM 28.20.

This book represents a considerably extended version of the second (Czech) edition (1952) of the author's *Introduction into Group Theory*, Přírodovědecké Vydavatelství, Prague; MR **15**, 7]. It contains three chapters: I. Sets (pp. 1–73), II. Groupoids (pp. 75–129), III. Groups (pp. 131–181). Chapter I is very different from the set-theoretical introductions of other books on algebraic structures. After the basic definitions ($\cap$, $\cup$, $\times$, $\subset$) there follows an extensive theory of partitions "in" and "on" (of) a set G (Zerlegungen in und auf einer Menge). A partition "in" G is a non-empty set of non-empty mutually disjoint subsets $\bar{a}$ of G; "on" G, if every $a \in G$ is an element of (exactly) one $\bar{a} \in \bar{A}$. If $\bar{A}$ is a partition in G and $B \subset G$, then the closure (Hülle) of B in $\bar{A}$, denoted by $B \sqsubset \bar{A}$ (or $\bar{A} \sqsupset B$) is defined as the set of all elements $\bar{a}$ of $\bar{A}$ which contain elements of B. Further the penetration (or intersection, Durchdringung) of the partition $\bar{A}$ by the subset B, denoted by $\bar{A} \sqcap B$ (or $B \sqcap \bar{A}$) is the set of all intersections $\bar{a} \cap B$ ($\bar{a} \in \bar{A}$). These operational symbols, added to those in common use, enable the author to formulate the algebra of sets in conjunction with an algebra of partitions which turns out to be useful in chapters II and III. It includes notions such as coverings, refinements, chains of partitions, complementary partitions. This is followed by a section on mappings of sets and partitions. As a simple example permutations are dealt with. Finally come multivalued mappings and series of partitions. The chapter ends with remarks on "scientific classifications".

A groupoid $\mathfrak{G}$ is defined as a set G with a composition ab, without any restriction on associativity of the "product": a mapping $G \times G \to G$. An algebra of subsets is developed, subgroupoids and ideals ($GA \subset A$) are defined, and also homomorphism (called "deformation"). A generating partition in G is an $\bar{A}$ such that for every $\bar{a}, \bar{b} \in \bar{A}$ there is a $\bar{c} \in \bar{A}$ such that $\bar{a}\bar{b} \subset \bar{c}$. If $\bar{A}$ is a partition on G, then $\bar{A}$ defines the factoroid $\mathfrak{A}$, i.e., a groupoid where the multiplication is defined as $\bar{a} \circ \bar{b} = \bar{c}$ if $\bar{a}\bar{b} \subset \bar{c}$. This is one of the most important ideas of the theory. It enables the author to generalize the isomorphism theorems of group theory, whereby the factoroid takes over the role of the factor group of group theory. In a section "special groupoids" there is a discussion of associative groupoids (semigroups, regular semigroups, groups) and also Brandt groupoids (which are not groupoids in the sense of the author) and groupoids with two compositions, which include lattices. In the last chapter the general results of chapter II are systematically specialized for groups. As examples the following relations may be mentioned: Let $\mathfrak{B} \subset \mathfrak{A}$, and $\mathfrak{C}$ be subgroups of $\mathfrak{G}$; if $\mathfrak{G}/_l\mathfrak{A}$ represents the system of the left cosets of $\mathfrak{G}$ (mod $\mathfrak{A}$), then $\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C})$; and if $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} = \mathfrak{B}(\mathfrak{A} \cap \mathfrak{C})$, then $\mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{B}$. After a restatement of the isomorphism theorems for groups, the last section gives, maybe a little late, an elementary discussion on cyclic groups. The work is written in the style of a textbook; every section contains problems, mostly not difficult. Thus the book is suitable for students. It ends with a list of 91 papers on partitions in sets and on equivalence, published in the years 1937–1958, and a list of 20 (mostly recent) books on group theory, including such works as those by Jacobson and van der Waerden. Algebraists will be grateful to the author for having made his work accessible to non-Czech readers.

H. Schwerdtfeger (Montreal)

8050:

Piccard, Sophie. Les éléments libres des groupes libres. C. R. Acad. Sci. Paris **251** (1960), 1328–1330.

Let G be the free group on a set A. Let $b \in G$ involve only the generators $a_1, a_2, \cdots, a_k \in A$. Then b is a possibly free generator of G if and only if it occurs in a set $b_1, b_2, \cdots, b_k$ obtained from $a_1, a_2, \cdots, a_k$ by Nielsen transformations.

P. J. Higgins (London)

8051:

Piccard, Sophie. Les groupes quasi libres. C. R. Acad. Sci. Paris **251** (1960), 2271–2273.

The author states without proof sixteen theorems about quasi-free groups, a concept she has defined in a previous note [same C. R. **250** (1960), 3260–3262; MR **22** #5666]. For example, the set of elements of a quasi-free group G which are of total degree zero in each element of a basis for G forms a subgroup which is the commutator subgroup of G. If a quasi-free group has a basis with m elements, then it has at least 2^m distinct normal subgroups, at least 2^m distinct bases, and more than m quasi-free subgroups. For every cardinal number m, there exists a quasi-free group with a basis of m elements, and with more than m outer automorphisms. For every integer $n \geqq 2$, every quasi-free group has a normal subgroup of index n. If a quasi-free group is abelian, it is a free abelian group.

O. Frink (Dublin)

8052:

Szep, J. Sui gruppi fattorizzabili. Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 20–30. Edizioni Cremonese, Rome, 1960.

Un groupe G est dit factorisable s'il est $G=AB$, où A et B sont des sous-groupes propres de G. L'auteur donne une exposition claire et complète de tous les principaux résultats connus sur les groupes factorisables. Il y a aussi une bonne bibliographie sur l'argument, contenant 83 articles.

G. Zappa (Florence)

8053:

Chen, Chung-mu; Tin, Min-yung. On the definition of a group. Advancement in Math. **4** (1958), 127–131. (Chinese)

The first part of this paper suggests a good idea for defining an abelian group. Let G be a non-empty set of elements $a, b, c, \cdots$ for which the product (ab) of any two elements a and b of G is defined. Under the operation, if any one of the following class of postulates holds, then the set G forms an abelian group:

I. (1) $ab \in G$ for all a, b and $c \in G$;
(2) $a(bc)=(ac)b$ for all a, b and $c \in G$;
(3) $ax=b$ and $ya=b$ have a unique solution in G, if $a, b \in G$.

II. (1) and (2) as in I. (1) and I. (2);
(4) $ax=b$ has a unique solution in G, for $a, b \in G$.

III. (1) as in I. (1);
(2′) $a(bc)=(ca)b$ for all $a, b, c \in G$;
(5) $ya=b$ has a unique solution in G, for $a, b \in G$.

IV. (1) and (5) as in III;
(2″) $a(bc)=(ba)c$ for all $a, b, c \in G$.

The second part gives a class of weaker conditions which replace the usual ones for defining a group: (1) if (ab) has meaning, then $ab \in G$; (2) if $(ab)c$ and $a(bc)$ have meaning, then $(ab)c=a(bc)$; (3) $ax=b$, $ya=b$ have a unique solution in G, for all a, b in G.

Ching-ju Chang (Taipei)

8054:

Dlab, Vlastimil. On cyclic groups. Czechoslovak Math. J. **10** (85) (1960), 244–254. (Russian summary)

Sharpening a theorem of F. Szász [Acta Sci. Math. Szeged **17** (1956), 83–84; MR **18**, 376], the author proves the following result: Let G be a group such that there exist integers $m_1, \cdots, m_k$ for which the m_ith powers $\{G^{m_i}\}$ are cyclic subgroups $(i=1, \cdots, k)$. Then the subgroup $\{G^t\}$ is cyclic too, if t is the g.c.d. of the m_i's. If the m_i's are relatively prime, then G is cyclic. From the above result there follows that G is cyclic if and only if every cyclic subgroup of G is a power $\{G^m\}$ of this group for a suitable integer [cf. F. Szász, Acta Math. Acad. Sci. Hungar. **6** (1955), 475–477; MR **17**, 940].

A. Kertész (Debrecen)

8055:

Charles, Bernard. Étude sur les sous-groupes d'un groupe abélien. Bull. Soc. Math. France **88** (1960), 217–227.

Soit G un groupe abélien et H un sous-groupe de G. L'auteur recherche s'il existe des sous-groupes purs minimaux contenant H. Il démontre d'abord que le cas général se ramène à celui des groupes abéliens primaires. Il démontre que si G est primaire et H contient des éléments de hauteur infinie, il n'existe pas en general des sous-groupes purs minimaux contenant H. Ensuite l'auteur démontre que si G est primaire et H est sans éléments de hauteur infinie, il existe un sous-groupe pur de G contenant H et sans élément de hauteur infinie. Le problème est ramené ainsi au cas où G est primaire sans élément de hauteur infinie. Après avoir examiné particulièrement le cas où H est cyclique, l'auteur donne le résultat suivant: Soit G un groupe abélien sans élément de hauteur infinie, et H un sous-groupe de G; il existe des sous-groupes purs minimaux de G contenant H dans les cas suivants: (1) H est contenu dans le socle de G; (2) H contient un sous-groupe K qui est pur dans G et dense dans H pour la topologie définie par les p^nG.

G. Zappa (Florence)

8056:

Mal'cev, A. I. On free soluble groups. Dokl. Akad. Nauk SSSR **130** (1960), 495–498 (Russian); translated as Soviet Math. Dokl. **1**, 65–68.

Auslander and Lyndon [Amer. J. Math. **77** (1955), 929–931; MR **17**, 709] have proved that if F is a free group and F/K is infinite, then $F/[K, K]$ has a trivial centre. The author uses this result to prove that if F/K is torsion-free, then any two commuting elements of $F/[K, K]$ are either contained in $K/[K, K]$ or are powers of one and the same element. From this it follows, in particular, that if an element of a free soluble group commutes with all its conjugates, then it lies in the last non-trivial term of the derived series. The author proves next that if F/K is an R-group (extraction of roots, if possible, is unique), then $F/[K, K]$ is also an R-group. This has as a consequence that all free soluble groups are R-groups. On the basis of these results, the author then proves that the elementary theory in the sense of Tarski [see Tarski, Mostowski and Robinson, *Undecidable theories*, North-Holland, Amsterdam, 1953; MR **15**, 384] of an n-step free soluble group with t generators is insoluble for $n \geqq 2$, $t \geqq 2$.

K. A. Hirsch (St. Louis, Mo.)

8057:

Судзуки, М. [Suzuki, Michio]. ★Строение группы и строение структуры ее подгрупп [Structure of a group and the structure of its lattice of subgroups]. Translated from the English by L. E. Sadovskiĭ; edited by B. I. Plotkin. Izdat. Inostr. Lit., Moscow, 1960. 158 pp. 5.20 r.

For original [Springer, Berlin, 1956] see MR **18**, 715. The present edition contains a preface and footnotes by

the editor, and the bibliography has been supplemented by recent literature.

8058:

Magnus, W. Some finite groups with geometrical properties. Proc. Sympos. Pure Math., Vol. 1, pp. 56–63. American Mathematical Society, Providence, R.I., 1959.

In certain groups the author defines a geometry by identifying certain conjugate classes of subgroups with geometrical elements and using the inner automorphisms of the group to define "motions" in the geometry. Specifically a group M defines a geometry and a group of motions in this geometry if it has the following properties: (i) There exists a complete conjugate set of proper subgroups P_i such that each P_i is its own normalizer in M. (ii) Calling P_i the zero-dimensional subgroups, d-dimensional subgroups for $d=1, 2, \cdots, n-1$ are defined recursively as follows. A d-dimensional subgroup is the intersection of a zero-dimensional subgroup P and a $d-1$ dimensional subgroup not contained in P. Any two d-dimensional subgroups contained in a $(d-1)$-dimensional subgroup Q are conjugate in Q. (iii) All d-dimensional subgroups are conjugate in M. (iv) The unit element is the only $(n-1)$-dimensional subgroup, but no d-dimensional subgroup, for $d<n-1$, consists only of the unit element.

If a group M satisfies (i), (ii), (iii) and (iv), a geometry Γ is defined as follows. The d-dimensional subgroups of M are called d-dimensional subspaces of Γ (except for $d=n-1$). Each inner automorphism t of M defines a motion of Γ which carries the d-dimensional subspace L into $L^*=tLt^{-1}$.

The motions induced by M in Γ satisfy the following. If L_d and $L_d{}^*$ are any two d-dimensional subspaces there is at least one motion mapping L_d into $L_d{}^*$. If L_{d+k} and L^*_{d+k} are two $(d+k)$-dimensional subspaces incident on a d-dimensional subspace L_d (i.e., L_{d+k} and L^*_{d+k} are both subgroups of L_d) there is at least one motion which keeps L_d fixed and maps L_{d+k} into L^*_{d+k}. Any motion leaving n points fixed which are not on an $(n-2)$-dimensional subspace is the identity. Γ has the geometric property that any set of $d+1$ points not incident on a $(d-1)$-dimensional subspace lie on exactly one d-dimensional subspace.

In this development $(n-1)$-dimensional subspaces do not exist in general. However, in some cases these can be defined if the group M contains subgroups which induce in Γ motions which have the usual geometric properties of translations.

A finite group of two-dimensional motions has the following two properties. (1) No extra postulates are needed to assure the existence of a subgroup of translations, which enables one to construct lines in Γ. (2) The group M defines Γ uniquely.

For three-dimensional groups M, the situation is more flexible. In general translations do not exist (and hence planes cannot be defined), and also two different geometries Γ_1 and Γ_2 may belong to the same group. The author quotes a result of Bachman which states necessary and sufficient conditions in order that a three-dimensional finite group M possess a subgroup of translations.

Finally, for $n>2$, the author remarks that a large number of simple groups may be interpreted as groups of n-dimensional motions. In particular $LF(m, g)$ in a Galois field of order g and the alternating groups may be so interpreted. *N. S. Mendelsohn* (Winnipeg, Man.)

8059:

Howarth, J. C. On the power of a prime dividing the order of the automorphism group of a finite group. Proc. Glasgow Math. Assoc. 4, 163–170 (1960).

The author studies the divisibility of the order of the automorphism group of a finite group G by the powers of a prime factor of the order of G, and refines the result due to J. A. Green [Proc. Roy. Soc. Ser. A **237** (1956), 574–581; MR 18, 464] as follows: For $h\geqq 12$, write $f(h)=(h+3)/2$ for odd h and $=(h+4)/2$ for even h. Then, if $p^{f(h)}$ divides the order of a group G, p^h divides the order of its automorphism group. The main tool is a refinement of the procedure of Green, by using a different lower bound of the order of the automorphism group of G which reduces to the identity on the factor group G/Z by its center Z. *O. Nagai* (*Yamaguchi*)

8060:

Grün, Otto. Einige Sätze über Automorphismen abelscher p-Gruppen. Abh. Math. Sem. Univ. Hamburg **24** (1960), 54–58.

Das Hauptergebnis der Arbeit lautet: Es sei A eine endliche abelsche p-Gruppe und $A=A_1\times A_2\times\cdots\times A_n$, wobei A_i vom Typus $(p^i, \cdots, p^i)$ $(i=1, \cdots, n)$ ist. Ist dann U eine Untergruppe mit zu p primer Ordnung der Automorphismengruppe von A, so können die direkten Faktoren A_i so gewählt werden, daß sie bei U einzeln invariant sind. *R. Kochendörffer* (Rostock)

8061:

de Vries, H.; de Miranda, A. B. Groups with a small number of automorphisms. Math. Z. **68** (1958), 450–464.

The paper is concerned with groups G which have at most eight automorphisms. The automorphism group of G is denoted by $A(G)$, and the order of G by $|G|$. The following results are proved.

Let $|A(G)|\leqq 8$. If G is finite non-abelian, then G is a dihedral group of order 6 or 8, and $A(G)\cong G$. If G is infinite non-abelian, then $A(G)$ is elementary abelian of order 8. If G is mixed abelian, then the periodic part of G has order 2, and $A(G)$ is elementary abelian of order 4 or 8; both these orders actually occur. For each group A of order at most 8, a complete list of the periodic abelian groups G with $A(G)\cong A$ is given; the orders of these G's do not accede 30.

For the rest, let G be torsion-free abelian. If $|A(G)|\leqq 8$ and G is decomposable, then $A(G)$ is non-cyclic abelian and has order 2^n with $n=2$ or 3; there are such G's of every rank r between n and $\aleph$. For each such $A(G)$ and r, there exist indecomposable G's as well. If $A(G)$ is cyclic of order 4, then G is (indecomposable) of infinite or finite even rank, and every such rank occurs up to $\aleph$. J. de Groot [Nederl. Acad. Wetensch. Proc. Ser. A **60** (1957), 137–145; MR 18, 790] has constructed, for each r between 1 and $\aleph$, (indecomposable) groups G of rank r with $|A(G)|=2$. Finally, if G is indecomposable and $A(G)$ is of order at most 8 but different from the groups mentioned so far, then $A(G)$ is either cyclic of order 6 or it is the quaternion group; both cases do occur.

At one point the paper goes beyond its program: It is shown that if G is torsion-free abelian and $A(G)$ is cyclic, then $|A(G)|$ cannot be divisible by 8.

{The authors have kindly called the attention of the reviewer to a minor slip in the proof of theorem 4, which can be put right by changing the definition of ψ (at the end of the second paragraph of the proof) to $\psi:\psi a_n=-a_n$, $\psi b_n=2a_n+b_n$. Further, one of the steps in the proof of theorem 7 is obtained by using a faulty result of Schenkman, but it can be obtained by a straightforward argument as well.} *L. G. Kovács* (Manchester)

8062a:

Chehata, C. G. An embedding theorem for groups. Proc. Glasgow Math. Assoc. **4**, 140–143 (1960).

8062b:

Chehata, C. G. Generalisation of an embedding theorem for groups. Proc. Glasgow Math. Assoc. **4**, 171–177 (1960).

Let G be a group. A homomorphism μ of a subgroup A of G into G is called a partial endomorphism of G. B. H. Neumann and Hanna Neumann [Proc. London Math. Soc. (3) **2** (1952), 337–348; MR **14**, 351] give necessary and sufficient conditions for a partial endomorphism of a group to be extendable to a (total) endomorphism of a supergroup. In a recent paper the author generalizes these results for the simultaneous extension of an arbitrary system of partial endomorphisms [Proc. Glasgow Math. Assoc. **2** (1954), 37–46; MR **16**, 10].

In the first of the two papers the following theorem is proved: The partial endomorphism μ of the group G which maps A onto B is extendable to a total endomorphism μ^* of a supergroup $G^*\supseteq G$ such that μ^* is an isomorphism on $G(\mu^*)^m$ (for some given positive m) if and only if there exists in G a sequence of normal subgroups $L_1\subseteq L_2\subseteq\cdots\subseteq L_m=L_{m+1}=\cdots$ such that $L_1\cap A$ is the kernel of μ, $(L_{i+1}\cap A)\mu=L_i\cap B$ $(i=1,2,\cdots,m)$.

The second paper considers a group G with a well-ordered set of partial endomorphisms $\mu(\alpha)$, and generalizing the result of the first paper gives necessary and sufficient conditions for the simultaneous extension of the $\mu(\alpha)$ to (total) endomorphisms $\mu^*(\alpha)$ of one and the same group $G^*\supseteq G$ such that $\mu^*(\alpha)$ is an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, where, for each α, $n(\alpha)$ is a positive integer. *A. Kertész* (Debrecen)

8063:

Wielandt, Helmut. Über den Transitivitätsgrad von Permutationsgruppen. Math. Z. **74** (1960), 297–298.

Bekanntlich wird vermutet, daß der Transitivitätsgrad einer Permutationsgruppe G des Grades n, die weder die alternierende noch die symmetrische Gruppe ist, eine von n unabhängige Schranke t $(\geqq 5)$ nicht überschreiten kann. Als wesentlicher Beitrag zu dieser Vermutung wird bewiesen: Falls die äußere Automorphismengruppe jeder einfachen Untergruppe von G auflösbar ist, kann G höchstens 7-fach transitiv sein. Sollte sich die Vermutung bestätigen, daß die äußeren Automorphismengruppen aller endlichen einfachen Gruppen auflösbar sind, so wäre auch die eingangs erwähnte Vermutung mit $t=7$ richtig. *R. Kochendörffer* (Rostock)

8064:

Itô, Noboru. Zur Theorie der Permutationsgruppen vom Grad p. Math. Z. **74** (1960), 299–301.

The author proves the following theorem: Let G be a non-soluble transitive permutation group of prime degree $p\geqq 5$; suppose that the normalizer of a Sylow p-subgroup of G has order $2p$; then p is a Fermat prime 2^m+1 and $G\cong SL(2, 2^m)$. The short but ingenious proof uses modular and ordinary character theory. *G. E. Wall* (Chicago, Ill.)

8065:

Wielandt, H. Arithmetische Struktur und Normalstruktur endlicher Gruppen. Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 56–65. Edizioni Cremonese, Rome, 1960.

A summary, without proofs, of the state, at the time of writing, of the relations between the normal structure of a finite group, and the existence and conjugacy of subgroups of certain orders, with related matter. *Graham Higman* (Chicago, Ill.)

8066:

Steinberg, Robert. Invariants of finite reflection groups. Canad. J. Math. **12** (1960), 616–618.

In complex affine n-space with a unitary metric, a "reflection" is a linear transformation whose invariant points are just the points of a hyperplane: the "mirror". Chevalley [Amer. J. Math. **77** (1955), 778–782; MR **17**, 345] proved that a finite group generated by such reflections possesses a set of n algebraically independent polynomial invariants which form a basis for the set of all invariants of the group. Let J denote the jacobian (matrix) of such a basic set of invariants, and P any point of the space. The author proves that the nullity of J at P is equal to: (a) the maximum number of linearly independent mirrors containing P; and (b) the maximum rank of $1-x$ for all matrices x in the group that leave P invariant. He has thus succeeded in proving a conjecture of Shephard [Enseignement Math. (2) **2** (1956), 42–48; MR **18**, 191]. *H. S. M. Coxeter* (Toronto)

8067:

Wielandt, Helmut. Beziehungen zwischen den Fixpunktzahlen von Automorphismengruppen einer endlichen Gruppe. Math. Z. **73** (1960), 146–158.

If $\mathfrak{G}$ and $\mathfrak{H}$ are finite groups, the group $\mathfrak{G}$ is said to be represented on the group $\mathfrak{H}$ if there corresponds, to every pair of elements $G\in\mathfrak{G}$ and $H\in\mathfrak{H}$, a uniquely determined element $H^G\in\mathfrak{H}$ in such a way that $(H_1H_2)^G=H_1{}^GH_2{}^G$, $H^{(G_1G_2)}=(H^{G_1})^{G_2}$, $H^I=H$. A subset $\mathfrak{K}$ of $\mathfrak{H}$ is called fixed by $\mathfrak{G}$ if $K\in\mathfrak{K}\Rightarrow K^G\in\mathfrak{K}$ for $G\in\mathfrak{G}$. An element which is fixed by $\mathfrak{G}$ is called a fixed point of $\mathfrak{G}$. The fixed points of $\mathfrak{G}$ in $\mathfrak{H}$ form a subgroup $\mathfrak{H}_{\mathfrak{G}}$. The number of fixed points of $\mathfrak{H}$ by $\mathfrak{G}$ is then the order $|\mathfrak{H}_{\mathfrak{G}}|$ of $\mathfrak{H}_{\mathfrak{G}}$. Generalizing and more deeply penetrating a result of Richard Brauer, the author gives algebraic relations between the fixed point numbers of subgroups $\mathfrak{G}_\nu$ of $\mathfrak{G}$ which have "prime" representations on $\mathfrak{H}$ in the following meaning: (1) The order of the group $\mathfrak{G}_\nu{}^*$ of automorphisms which is induced by $\mathfrak{G}_\nu$ and the order of $\mathfrak{H}$ are relative prime. (2) Every subgroup of $\mathfrak{H}$ which is fixed by $\mathfrak{G}_\nu$ has at least one p-Sylow-group for every prime number p which is fixed by $\mathfrak{G}_\nu$. The algebraic

relations are closely connected with relations of the form

$$|\mathfrak{H}_{\mathfrak{G}}|^{g}=\prod_{\mathfrak{Z}\in\mathfrak{z}}|\mathfrak{H}_{\mathfrak{Z}}|^{\mu(\mathfrak{G},\mathfrak{Z})z},$$

where $g=|\mathfrak{G}|$, $z=|\mathfrak{Z}|$, $\mu(\mathfrak{G},\mathfrak{Z})$ is an integer which is obtained by means of the Möbius function, and $\mathfrak{z}$ is the set of cyclic subgroups of $\mathfrak{G}$. *H. Bergström* (Göteborg)

8068:

Safonov, S. A. Groups with a single class of inaccessible iso-ordinal Πd-subgroups. Dokl. Akad. Nauk SSSR **130** (1960), 26–28 (Russian); translated as Soviet Math. Dokl. **1**, 16–18.

Soit Π un ensemble de nombres premiers. Un groupe fini, dont l'ordre est divisible pour quelque nombre premier contenu dans Π, est dit Πd-groupe. Un sous-groupe d'un groupe fini G est dit accessible s'il appartient à un serie normale de G; inaccessible dans le cas contraire. L'auteur démontre le théorème suivant. Si tous les Πd-sous-groupes inaccessible d'un groupe fini G ont le même ordre, G est résoluble. En particulier: Si tous les sous-groupes inaccessibles d'un groupe fini G ont le même ordre, G est résoluble. *G. Zappa* (Florence)

8069:

Thompson, John G. A special class of non-solvable groups. Math. Z. **72** (1959/60), 458–462.

If the finite non-soluble group G has a nilpotent maximal subgroup $P\times M_1$, where M_1 has odd order and P is generalized quaternion or dihedral, then G has a normal series $G\geq G_0>T\geq 1$, where T is nilpotent, $[G:G_0]\leq 2$, and $G_0/T\simeq \mathrm{LF}(2,q)$ where q is a prime $2^n\pm 1$ greater than 5, or $q=9$; and if $q=7$, $[G:G_0]=2$. *Graham Higman* (Chicago, Ill.)

8070:

Thompson, John G. Normal p-complements for finite groups. Math. Z. **72** (1959/60), 332–354.

It is proved that if G is a finite group, P a Sylow p-subgroup of G, where p is an odd prime, and $\mathfrak{A}$ a group of automorphisms of G leaving P fixed, then either G has a normal p-complement, or there is an $\mathfrak{A}$-invariant normal subgroup U of P such that the factor group $N(U)/C(U)$ of its normaliser by its centraliser is not a p-group. The proof of this important theorem is long and complicated, and the remarks which follow are intended to indicate the main ideas, rather than to be an orderly summary.

Naturally, the theorem is proved by induction on the order of G, and by contradiction; so we assume that $(G,P,\mathfrak{A})$ is a smallest counterexample. Then G has no normal p-complement, so that [e.g., M. Hall, *The theory of groups*, Macmillan, New York, 1959; MR **21** #1996] there is a subgroup H of P, not necessarily either normal or $\mathfrak{A}$-invariant, such that $N(H)/C(H)$ is not a p-group. Choose H maximal, in the sense that as high a power of p as possible divides the order of $N(H)$, and, subject to this, H itself is as large as possible. Suppose further that $P_1=P\cap N(H)$ is a Sylow p-subgroup of $N(H)$. The maximality of H implies that if $H<J<P_1$, then $N(J)/C(J)$ is a p-group, and this in turn, using the inductive hypothesis with $\mathfrak{A}$ trivial, implies that $N(H)/H$ has normal p-complement.

This fact is used to extract from $N(H)$ a subgroup K which preserves the awkwardness of $N(H)$ but has transparent structure. Specifically, K contains P_1, and has order $p^\alpha q^\beta$, where q is a prime, not p, dividing the order of $N(H)/C(H)$. A Sylow q-subgroup Q of K is not normal in K, but if Q_1 is the largest normal q-subgroup of K, Q/Q_1 is elementary abelian. Moreover, Q_1H/Q_1 is the largest normal p-subgroup of K/Q_1, and from this follows that q-elements of K which centralise H lie in Q_1. Because $N(H)/H$ has normal p-complement, QH is normal in K.

Now let $A\,(\neq 1)$ be an $\mathfrak{A}$-invariant normal subgroup of P, such that $[P,A,A]=1$. Using a theorem of P. Hall and the reviewer [Proc. London Math. Soc. (3) **6** (1956), 1–42; MR **17**, 344], the facts that $1\leq Q_1<Q_1H<QH\leq K$ is the upper p-series of K, that Q/Q_1 is abelian, and that $p\geq 3$, imply that if a subgroup X of P_1 is not contained in H, then $[H,X,X]\neq 1$. Thus $A\cap P_1$ $(=A\cap N(H))$ is contained in H; from this it follows that A itself is contained in H.

The rest of the proof (indeed, the whole proof) is more or less explicitly concerned with the weak closure $V(A,P)$ of A in P, that is, with the subgroup generated by the conjugates of A that lie in P. Consider then a conjugate C of A that lies in P_1. The p-group CH acts on the elementary abelian q-group QH/Q_1H; we choose R in Q so that RH/Q_1H is a minimal CH-invariant subgroup of QH/Q_1H. Suppose first that, for some A, and for all choices of C and R, Q_1CH is normal in RCH. Then for all C, Q_1CH is normalised by Q, whence Q_1VH is normalised by Q, where $V=V(A,P_1)$. It follows that $V\leq H$, i.e., that $V(A,H)=V(A,P_1)$. A new appeal to the maximality of H then shows that $P_1=P$, so that $V=V(A,H)=V(A,P)$. Because $V=V(A,P)$, V is a normal $\mathfrak{A}$-invariant subgroup of P. Because $V=V(A,H)$, Q normalises V. But Q does not centralise V. Indeed, Q does not centralise A, because, by an appeal to the minimality of G, $N(A)$ has a normal p-complement, so that if Q is contained in $N(A)$, Q centralises H, which is not so. Thus we only have to show that A exists satisfying $[P,A,A]=1$, and the condition of this paragraph.

If for some choice of A, C, and R, Q_1CH is not normal in RCH, let H_1 be the largest subgroup of P_1 such that Q_1H_1 is normal in RCH, so that $H\leq H_1<CH$. It is next shown that if W is the group consisting of the elements of order p in the centre of H_1, then RCH/Q_1H_1 acts faithfully on Q_1W/Q_1, and, again using the theorem of Hall and the reviewer, it follows that $[W,C,C]\neq 1$. The general idea behind the conclusion of the proof is to use this fact, and the fact that $[P^y,C,C]=1$, where $C=A^y$ (or some similar pair of contrasting facts), to give a contradiction in the presence of special information about A. For instance, we have such a contradiction if we can show that W normalises B^y, where B is an $\mathfrak{A}$-invariant normal subgroup of P. For then C lies in essentially different ways in two Sylow subgroups of $N(B^y)$, which is impossible, because $N(B^y)$ has a normal p-complement. The last stages of the argument in fact exclude first the possibility that A be abelian but not elementary, then that it be a maximal abelian normal subgroup of P. But if no maximal abelian normal subgroup of P is $\mathfrak{A}$-invariant, A can be chosen to be non-abelian, and to satisfy also certain specific conditions; and this is also shown to be incompatible with $H_1<CH$. *Graham Higman* (Chicago, Ill.)

8071:

Thompson, John. Finite groups with normal p-complements. Proc. Sympos. Pure Math., Vol. 1, pp. 1–3.

American Mathematical Society, Providence, R.I., 1959.

An informal account of the paper reviewed above.

Graham Higman (Chicago, Ill.)

8072:

Karrer, Guido. Über die Erweiterungen der Spindarstellungen. Ann. Acad. Sci. Fenn. Ser. A I No. 277 (1960), 11 pp.

Let $O_n^+(R, f)$ be the n-dimensional unimodular orthogonal group constructed from a field R which is not of characteristic 2 and in which sums of squares are always squares, and from a form f of arbitrary index; let $SL_n(R)$ be the corresponding unimodular group. If $n \geqq 2$ the author shows that the spin representation of $O_n^+(R, f)$ cannot be extended to a representation of $SL_n(R)$. The key to the proof is a study of the interplay among the unimodular transformations which are diagonal relative to some orthogonal basis of the underlying space.

R. Steinberg (Los Angeles, Calif.)

8073:

Srinivasan, Bhama. On the indecomposable representations of a certain class of groups. Proc. London Math. Soc. (3) **10** (1960), 497–513.

A study of the representation theory, mod p, of a finite group G with a series $1 \lhd H \lhd K \lhd G$, where H is of order prime to p, K/H is of order a power of p, and G/H is of order prime to p and cyclic. First, the author shows that every indecomposable G-module is isomorphic to a module obtained by the following recipe. Take a subgroup T of G, containing H. Take an indecomposable T/H-module E, and regard it as a T-module; and take a T-module S which is already irreducible as H-module. Form the tensor product $E \otimes S$ (as T-module), and, lastly, the induced G-module. Secondly, she studies the blocks of G, and obtains a formula for the number of irreducible G-modules with given defect group. Thirdly, she studies indecomposable G-modules with cyclic defect group. Such a module is a submodule of a principal G-module, and a principal G-module has cyclic defect group if and only if it has a unique composition series. Finally, the results are applied to the case of a p-soluble group G with cyclic Sylow p-subgroup.

Graham Higman (Chicago, Ill.)

8074:

Newman, M. F. On a class of metabelian groups. Proc. London Math. Soc. (3) **10** (1960), 354–364.

A just metabelian group, introduced by B. H. Neumann [Compositio Math. **13** (1956), 47–64; MR **19**, 632], is a group with non-trivial abelian commutator subgroup such that every proper homomorphic image is abelian. A just metabelian group with trivial centre is called briefly a JM-group. The paper investigates the structure of JM-groups. Let μ' be the mapping of the JM-group G into the automorphism group of the commutator group G', defined by $g\mu' = \gamma$ for all g in G, where γ is the restriction to G' of the inner automorphism of G induced by g. The subring Θ of the endomorphism ring $\mathfrak{E}$ of G' generated by the image Γ of μ' turns out to be a field, which is useful for describing JM-groups. The field Θ will be called the associated field of G and the group Γ the associated group of automorphisms of G.

The main results of the paper can be summarized as follows: The JM-group G is isomorphic to the group $L(\Gamma, \Theta)$ of all linear inhomogeneous substitutions $z \to \lambda z + \omega$, where $\lambda \in \Gamma$, $\omega \in \Theta$. A necessary and sufficient condition for a non-abelian group of linear inhomogeneous substitutions $L(\Lambda, \Omega)$, where Ω is a field and Λ a subgroup of the multiplicative group of Ω, to be a JM-group is that the additive closure Λ^+ of Λ in Ω be Ω. Two JM-groups are isomorphic if and only if there is an isomorphism between their associated fields which maps their respective associated groups of automorphisms onto one another. A finite JM-group is the extension of an elementary abelian group of order p^k (p is a prime, $p^k \neq 2$) by an automorphism of order n, where n divides $p^k - 1$ but not $p^j - 1$ for j less than k.

A. Kertész (Debrecen)

8075:

Schwarz, Štefan. On dual semigroups. Czechoslovak Math. J. **10** (85) (1960), 201–230. (Russian summary)

Let S be a semigroup with zero element. For any subset $A \subseteq S$ let $\mathscr{L}(A) = \{x \in S | xA = 0\}$, and $\mathscr{R}(A) = \{x \in S | Ax = 0\}$. Then S is called a dual semigroup if $\mathscr{L}[\mathscr{R}(L)] = L$ and $\mathscr{R}[\mathscr{L}(R)] = R$ for every left ideal L, and right ideal R of S. In this case the mappings $L \to \mathscr{R}(L)$ and $R \to \mathscr{L}(R)$ are inverse anti-isomorphisms of the complete lattices of left and right ideals of S. The main result of this paper is a theory of dual semigroups analogous to the known theory of dual rings as developed by I. Kaplansky [Ann. of Math. (2) **49** (1948), 689–701; MR **10**, 7]. First suppose S to be a dual semigroup without nilpotent ideals, such that every two-sided ideal of S contains a minimal two-sided ideal of S. Then $S = \bigcup_{\lambda \in \Lambda} M_\lambda$, where $M_\alpha \cdot M_\beta = M_\alpha \cap M_\beta = 0$ for $\alpha \neq \beta \in \Lambda$. The M_λ are minimal two-sided ideals of S, and each is a simple dual semigroup. A simple dual semigroup containing a minimal left and a minimal right ideal may be represented as the set of all $I \times I$ matrices (where I is some index set) with entries from a group with zero adjoined, in which at most one element in each matrix is different from zero. Conversely such matrix semigroups are dual and contain minimal one-sided ideals.

The radical of a semigroup S is defined as the set-theoretical union of its nilpotent left ideals, and is a two-sided ideal which need not itself be nilpotent. If the radical N of a dual semigroup S is nilpotent, such radical is also, under suitable additional restrictions, the intersection of all maximal left (right, two-sided) ideals of S; and the difference semigroup S/N is shown to be a dual semigroup without nilpotent ideals.

In the final section, the effect of the existence of a unit element on the structure of a dual semigroup is examined. It is shown, for example, that a dual semigroup without nilpotent ideals contains a unit element if and only if it is a group with zero adjoined.

K. G. Wolfson (New Brunswick, N.J.)

8076:

Artzy, R. Relations between loop identities. Proc. Amer. Math. Soc. **11** (1960), 847–851.

It is well known [Pickert, *Projective Ebenen*, Springer, Berlin, 1955; MR **17**, 399] that every loop L can be represented as a loop of a 3-net. Denote a 3-net N by (A, B, C)-net. Clearly, the interchange of line families leads to the definition of other loops defined on the same elements. The author considers the 3-nets (B, C, A) and (A, C, B) and determines consequences of imposing relations between the operations in the various loops. New proofs

of well-known results are obtained; for example, "In a weak inverse loop the right, middle and left nuclei coincide." *L. J. Paige* (Los Angeles, Calif.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 8014, 8103, 8367, 8375, 8520.

8077:

Berezin, F. A. The Laplacian operators on semisimple Lie groups and on some symmetric spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 364–369.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 152–156 [MR **19**, 292].

8078:

Urbanik, K. Remarks on compactly generated Abelian topological groups. Colloq. Math. **7** (1959/60), 187–190.

Let G be a locally compact abelian group. If C is a compact subset of the dual group $\hat{G}$, $\mathscr{F}(C)$ is defined to be the set of all functions in $L^2(G)$ whose Fourier transforms vanish off C. Each function in $\mathscr{F}(C)$ is equivalent to a continuous function. G is said to have property (H) if for each compact C in $\hat{G}$ there is a finitely generated algebraic subgroup D of G so that each function in $\mathscr{F}(C)$ is determined by its values on D. Answering a question raised by S. Hartman, the author proves that G is compactly generated if and only if it has property (H).

K. deLeeuw (Princeton, N.J.)

8079:

Ullrich, M.; Urbanik, K. A limit theorem for random variables in compact topological groups. Colloq. Math. **7** (1959/60), 191–198.

Let G be a compact topological group, λ a probability measure on G, that is, positive on each non-empty open subset of G. Let $\{a_n\}$ be a sequence with $0 \leq a_n \leq 1$ and $\sum a_n = \infty$. The authors show that if $\{\mu_n\}$ is a sequence of symmetric probability measures on G with $\mu_n(E) \geq a_n\lambda(E)$ for all Borel subsets E of G, then the convolutions $\mu_1 * \cdots * \mu_n$ converge, in the weak* topology of measures, to some multiple of Haar measure on G. They show, moreover, that the condition $\sum a_n = \infty$ cannot be weakened.

K. deLeeuw (Princeton, N.J.)

8080:

Borel, Armand. Commutative subgroups and torsion in compact Lie groups. Bull. Amer. Math. Soc. **66** (1960), 285–288.

Various results on the non-occurrence of p-torsion in a compact Lie group are proved. In particular, new results on the torsion of the exceptional Lie groups are obtained. An earlier conjecture of the author is settled, viz., a simple and simply connected Lie group has no p-torsion if p does not divide the coefficients of the linear combination of simple roots which expresses the highest root.

W. T. van Est (Leiden)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

8081:

Cohen, Haskell. A clan with zero without the fixed point property. Proc. Amer. Math. Soc. **11** (1960), 937–939.

An example is given of a compact, connected, topological semigroup with identity and zero such that the semigroup does not have the fixed point property.

Anne Lester (New Orleans, La.)

8082:

Koch, R. J. Ordered semigroups in partially ordered semigroups. Pacific J. Math. **10** (1960), 1333–1336.

A local semigroup is a Hausdorff space, S, with an open subset U of S and a multiplication $m: U \times U \rightarrow S$ which is continuous and associative insofar as is meaningful. A unit is an element u in U such that $ux = x = xu$ for all $x \in U$. A local semigroup S is said to be partially ordered if the relation $\leq$, defined by $a \leq b$ if and only if $a = bc$, is reflexive and antisymmetric. The author proves the following theorem: Let S be a locally compact partially ordered local semigroup with unit u, and let U_0 be a non-degenerate open connected set about u with $U_0{}^6$ defined; then S contains a non-degenerate compact connected linearly ordered local sub-semigroup L with $u \in L \subset U_0$. The following result is obtained as a corollary, where a standard thread is a compact connected semigroup irreducibly connected between a zero and a unit: If S is a non-degenerate compact connected partially ordered semigroup with unit u, then the minimal ideal K consists of left zeros for S, K consists of the set of minimal elements, and some elements of K can be joined by a standard thread to the unit.

Anne Lester (New Orleans, La.)

8083:

Fleischer, Isidore. Über Dualität in lineartopologischen Moduln. Math. Z. **72** (1959/60), 439–445.

Dans ces dernières années, la notion de module injectif a permis de placer la théorie de la dualité dans un cadre plus général englobant les divers cas particuliers antérieurement étudiés: si M est un R-module (à gauche), on considère non le module des homomorphismes de M dans R_S, mais $\mathrm{Hom}_R(M, I)$, où I est un R-module à gauche injectif; si $\bar{R} = \mathrm{End}_R(I)$, $M^* = \mathrm{Hom}_R(M, I)$ est un $\bar{R}$-module à gauche. L'auteur considère le cas où M est muni d'une topologie linéaire (c'est-à-dire ayant un système fondamental de voisinages de 0 qui sont des sous-modules) et séparée; on considère alors le sous-module N de M^* formé des homomorphismes continus de M dans I (I étant toujours muni de la topologie discrète). On peut munir N de topologies linéaires, comprises entre une topologie faible et une topologie forte, comme dans le cas particulier des espaces vectoriels (à condition de supposer que I contienne une image non réduite à 0 de tout R-module monogène); lorsque $\bar{R}$ est un anneau topologique, on a une notion de bidual et de réflexivité. L'auteur montre que les propriétés générales relatives à ces notions redonnent divers résultats particuliers de Kaplansky, Leptin et Schöneborn sur les modules linéairement compacts.

J. Dieudonné (Paris)

FUNCTIONS OF REAL VARIABLES

8084:

Hewitt, Edwin. ★Theory of functions of a real variable. With the editorial assistance of Kenneth A. Ross. Preliminary edition. Holt, Rinehart and Winston, New York, 1960. viii+326 pp. $4.00.

As the author states in his introduction, this book is a preliminary edition of a text planned for letterpress publication in about 1965. It has grown from lectures over a number of years, and is designed to cover a first-year graduate course. The 318 pages of typewritten text are distributed over four chapters: I. Preliminaries (49 pp.); II. Measures on the line (62 pp.); III. Lebesgue integral (92 pp.); IV. Lebesgue-Stieltjes integrals (115 pp.).

The first chapter is devoted to an introduction in set theory, relations and functions, cardinal numbers, and partial ordering (with a proof of Zermelo's well-ordering theorem and Zorn's lemma), and the final sections contain a discussion of the algebraic and topological structure of the real line.

In the second chapter, after some general remarks on linear functionals and measures, the theory of Lebesgue measure in the real line is developed, following the usual Carathéodory extension procedure. Banach's proof of Vitali's covering theorem is included, and the chapter ends with remarks on finitely additive measures defined for all subsets of the real line. In order to show the existence of such measures the Hahn-Banach theorem is proved first.

The first part of the third chapter contains an account of Lebesgue integration with respect to an abstract measure in an arbitrary point set. The following definition is chosen: If μ is a measure in the point set X, $f(x)$ a μ-measurable non-negative function on X, and $X=\sum_{k=1}^{n} X_k$ a decomposition of X into μ-measurable sets $X_1, \cdots, X_n$ with $M_k=\inf f$ on X_k, then the Lebesgue integral of f over X is the supremum of $\sum_{k=1}^{n} M_k\mu(X_k)$ for all such decompositions. The usual theorems are derived, and special attention is paid to the "ordinary" Lebesgue integral over the real line (comparison with the Riemann integral, differentiation theory, application to Fourier sine series with monotonely decreasing coefficients). The final sections are devoted to L_p spaces. In connection with the space L_2 there is a fairly extensive discussion of abstract Hilbert space.

The last chapter starts with a long section (the longest of the book, 38 pp.) dealing with F. Riesz's representation theorem for bounded linear functionals on the continuous functions with compact support. The discussion is restricted to functions on the real line but can be carried over word for word to functions on any locally compact Hausdorff space. In the next section it is shown how the measures so obtained in the real line are related to monotone point functions. Subsequent sections deal with the Radon-Nikodym theorem (Lebesgue decomposition of a measure with respect to another measure), the Jordan and Hahn decompositions of a set function, Fubini's theorem (for σ-finite measures), and finally the theorem that the conjugate space of L_p $(1\leq p<\infty)$ is L_q, $p^{-1}+q^{-1}=1$.

A great number of examples is included in the text, and more examples and some extensions of the theory may be found in the exercises. An index and a list of symbols facilitate the checking of earlier results.

The book is written in a lucid style, and is a valuable addition to the increasing number of modern textbooks combining integration theory and a first introduction to functional analysis. Perhaps its value could still be increased if in a subsequent edition it were shown more clearly that the Riesz representation theorem for positive linear functionals on the continuous functions with compact support is not just a lucky coincidence; the existence of a measure such that the functional can be written as an integral with respect to this measure is due essentially to the fact that the linear vector lattice C of continuous functions with compact support has the property that $f\in C$ implies $\min(f, 1)\in C$ (or, equivalently, $f\in C$ implies $f^2\in C$). More generally, the Riesz representation theorem is a particular case of an abstract representation theorem which has nothing to do with topological spaces. Finally, but this is of minor importance, the present reviewer has not been able to detect what, in the author's terminology, the difference is between a Lebesgue integral and a Lebesgue-Stieltjes integral.

A. C. Zaanen (Pasadena, Calif.)

8085:

Olmsted, John M. H. ★Real variables: an introduction to the theory of functions. The Appleton-Century Mathematics Series. Appleton-Century-Crofts, Inc., New York, 1959. xvi+621 pp. $9.00.

This book is designed to take a student with one year of calculus through an axiomatic treatment of real numbers, a detailed study of functions of one, two, and several variables, and an extension to functions with abstract metric spaces for domain and range. The definition of function is in the tradition of Dirichlet. The symbol $f(x)$ stands for the value of the function at the point x and for the function itself. If D and R are two sets of objects then $y=f(x)$ is called a function with domain D and range R if to each member x of D there corresponds at least one member y of R. Thus $y=\arcsin x$ is a function. This is contrary to what has become universal practice, which is to say that $y=\arcsin x$ is a relation which defines an infinite number of functions.

A range of topics wider than that usually found in books on advanced calculus is covered. In an exercise there is a proof of the Lebesgue theorem on dominated convergence for the Riemann integral. Functions of bounded variation and Riemann-Stieltjes integrals are considered. The chapter on Fourier series includes a study of linear function spaces, distance, inner product and orthogonality. Fejér's theorem is included.

There are many exercises, over 2200! The book is, without question, a valuable contribution to the field of instruction in analysis. *R. L. Jeffery* (Wolfville, N.S.)

8086:

Hahn, O.; Ellers, E.; Dzewas, J. Eine Einführung in die Analysis mittels des Filterbegriffs. I. Math. Naturwiss. Unterricht **13** (1960/61), 433–448.

An introduction to the theory of limits and continuity is based on the notion of filter. The exposition is intended for the pre-calculus level student and is preceded by an elementary introduction to set theory. The term "filter" as used here means "base of a filter" in the sense of Bourbaki. *J. C. Oxtoby* (Bryn Mawr, Pa.)

8087:

Costinescu, Olga. Sur le second théorème de la moyenne (le cas des intégrales doubles). An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 13–22. (Russian and Romanian summaries)

In this paper the author considers how one can extend the second mean value theorem to double integrals. A simple result in this direction is formulated and proved. This result is then utilized to obtain a criterion for the existence of an (improper) Riemann double integral over a rectangle of a function $f(x, y)$ which has one singular point. Unfortunately the condition obtained (theorem 3) is only a necessary condition; it is not sufficient unless $f(x, y)$ has constant sign near the singularity. [For example, consider the function $f(x, y) = 1/x$, and the region of integrability, a square of side 1 and center the origin.]

S. J. Taylor (Ithaca, New York)

8088:

Kulbacka, Marie. Sur l'ensemble des points de l'asymétrie approximative. Acta Sci. Math. Szeged **21** (1960), 90–95.

Let f be a real-valued function on the real line. A real number a is called a right [resp., left] approximate limit value of f at a point x_0 if for every positive ε the set of all x greater [resp., less] than a, for which $|f(x)-a| < \varepsilon$, has positive upper density at x_0. Let $W^+(x_0)$ [resp., $W^-(x_0)$] denote the set of all right [resp., left] approximate limit values of f at x_0. It is shown that the set of all x_0 for which $W^+(x_0) \neq W^-(x_0)$ is of first category and also (in accord with a remark of A. Császár) of measure zero.

T. A. Botts (Charlottesville, Va.)

8089:

Ganguli, P. L. Some results on a class of non-absolutely continuous functions. Bull. Calcutta Math. Soc. **51** (1959), 135–141.

Further study of a certain class [see Sengupta and Ganguli, same Bull. **51** (1959), 53–56; MR **22** #2664] of strictly isotone, continuous, non-absolutely-continuous real-valued functions on the unit interval.

T. A. Botts (Charlottesville, Va.)

8090:

Specht, Wilhelm. Zur Theorie der elementaren Mittel. Math. Z. **74** (1960), 91–98.

Let $\mathfrak{P}_n$ be the set of all n-dimensional vectors $x = (x_1, x_2, \cdots, x_n)$ with positive components, and $p = (p_1, p_2, \cdots, p_n)$, $p_1+p_2+\cdots+p_n = 1$, positive weights. The following quantities are defined for any vector $x \in \mathfrak{P}_n$: (1) $\mathfrak{M}_s(p, x) = (\sum p_\nu x_\nu^s)^{1/s}$, $s \neq 0$, real, the power mean, (2) $\mathfrak{M}_0(p, x) = \exp(\sum p_\nu \log x_\nu)$, the geometric mean, (3) $\mathfrak{M}_{-\infty}(p, x) = \lim_{s\to-\infty} (\sum p_\nu x_\nu^s)^{1/s} = \min(x_\nu)$, $\mathfrak{M}_{+\infty}(p, x) = \lim_{s\to+\infty} (\sum p_\nu x_\nu^s)^{1/s} = \max(x_\nu)$, the extrema, (4) $\beta(x) = \max(x_\nu)/\min(x_\nu)$, the relative width of the vector x, (5) B a constant satisfying the inequality $\beta(x) \leq B$, and (6) the quotient $Q_{s,t}(p, x) = \mathfrak{M}_t(p, x)/\mathfrak{M}_s(p, x)$, $-\infty < s < t < +\infty$. Theorem 1: For any vector $x \in \mathfrak{P}_n$ of the relative width $1 \leqq \beta(x) \leqq B$ and for any distribution of weights p, the inequality $1 \leqq Q_{s,t}(p, x) \leqq \Gamma_{s,t}(B)$, $-\infty < s < t < +\infty$ holds, where

$$(1)\quad \Gamma_{s,t} = \left(\frac{t-s}{t}\cdot\frac{B^t-1}{B^t-B^s}\right)^{1/s}\left(\frac{s}{t-s}\cdot\frac{B^t-B^s}{B^s-1}\right)^{1/t}, \text{ for } st \neq 0,$$

$$(2)\quad \Gamma_{0,t}(B) = \lim_{s\to 0}\Gamma_{s,t}(B) = \left(\frac{B^t-1}{t\log B}\right)^{1/t}\exp\left(\frac{\log B}{B^t-1}-\frac{1}{t}\right),$$

$$(3)\quad \Gamma_{s,0}(B) = \lim_{t\to 0}\Gamma_{s,t}(B) = \left(\frac{s\log B}{B^s-1}\right)^{1/s}\exp\left(\frac{1}{s}-\frac{\log B}{B^s-1}\right).$$

The $\Gamma_{s,t}(B)$ are the best bounds for the inequality. Theorem 2: If $p(x)$ denotes a non-negative integrable normed distribution function of the interval $0 \leqq x \leqq 1$ ($\int_0^1 p(x)dx = 1$), $f(x)$ a positive and integrable function in $0 \leqq x \leqq 1$, $\mathfrak{M}_s(p, f) = [\int_0^1 f^s(x)p(x)dx]^{1/s}$, $s \neq 0$, $\mathfrak{M}_0(p, f) = \exp[\int_0^1 \log f(x)\cdot p(x)dx]$ and if $0 < f_0 \leqq f(x) \leqq F_0$, then the inequality $1 \leqq \mathfrak{M}_t(p, f)/\mathfrak{M}_s(p, f) \leqq \Gamma_{s,t}(F_0/f_0)$ holds for $-\infty < s < t < +\infty$. Two examples of application are given.

J. W. Andrushkiw (Newark, N.J.)

8091:

Faĭnšmidt, V. L. On a class of regularly monotonic polynomials. Dokl. Akad. Nauk SSSR **130** (1960), 994–996 (Russian); translated as Soviet Math. Dokl. **1**, 134–136.

The author considers polynomials f which, with their first m derivatives, do not change sign on $[0, 1]$; suppose that $f(x)f'(x) \geqq 0$, and define "typical numbers" λ_k by $f^{(i-1)}(x)f^{(i)}(x) \geqq 0$ for $i \leqq \lambda_1$, $f^{(i-1)}(x)f^{(i)}(x) \leqq 0$ for $\lambda_1 < i \leqq \lambda_1+\lambda_2$, etc. The author considers several classes where λ_1 and λ_2 are both 0 or 1 and all other typical numbers are 2; he constructs the polynomials which deviate least from zero, with certain coefficients assigned.

R. P. Boas, Jr. (Evanston, Ill.)

8092:

Taylor, S. J. An alternative form of Egoroff's theorem. Fund. Math. **48** (1959/60), 169–174.

The author proves the following theorem. If a sequence $\{f_n(x)\}$ of measurable real-valued functions, defined on a Lebesgue measurable set E in Euclidean m-space, converges pointwise in E to a function $f(x)$, then there exists a decreasing sequence of real numbers $\delta_n \to 0$, and a set N of measure zero, such that $|f_n(x)-f(x)|/\delta_n \xrightarrow[n]{} 0$ for all $x \in E-N$. The proof is based on Egoroff's theorem and in turn gives a sharper form of Egoroff's theorem. The result is used to give stronger forms of the density theorem in m-space and the theorem on differentiating an absolutely continuous function of a real variable.

Y. N. Dowker (London)

8093:

Vul, E. B. Uniqueness theorems for a certain class of functions represented by integrals. Dokl. Akad. Nauk SSSR **129** (1959), 722–725. (Russian)

Soit $\sigma(\lambda)$ une fonction complexe, à variation bornée, telle que (1) $\int_{-\infty}^0 \exp[|\lambda|^{1/2}x]|d\sigma(\lambda)| < C\exp[h(x)]$ $(x > 0)$, où $h(x)$ est une fonction dérivable, non-décroissante, telle que $\lim_{x=\infty} xh'(x)/h(x) = \gamma > 1$. Soit g la fonction inverse de h'. Si $\int_1^\infty [g(t)/t^2]dt = \infty$, et si (2) $f(x) = \int\cos(\lambda x)^{1/2}d\sigma(\lambda)$, σ satisfaisant (1) et (2) est unique. Par contre si $\int_1^\infty [g(t)/t^2]dt < \infty$, il existe un f admettant plus d'un σ satisfaisant (1) et (2).

S. Mandelbrojt (Paris)

MEASURE AND INTEGRATION

See also 8084, 8232, 8320, B8555, B8556.

8094:

Cross, George E. The relation between two definite integrals. Proc. Amer. Math. Soc. **11** (1960), 578–579.

Where r is a positive integer James [Trans. Amer. Math. Soc. **76** (1954), 149–176; MR **15**, 611] has introduced, for real-valued functions of a real variable, a notion of P^{r+1}-integrability and has shown that P^{r+1}-integrability is implied by C_rP-integrability in the sense of Burkill [Proc. London Math. Soc. (2) **39** (1935), 541–552]. The present note determines a factor, dependent on r, whereby the C_rP- (definite) integral of a C_rP-integrable function f differs from the P^{r+1}- (definite) integral of f.

T. A. Botts (Charlottesville, Va.)

8095:

Lahiri, B. K. On sets under certain transformations. Bull. Calcutta Math. Soc. **51** (1959), 145–153.

Continuing an earlier study [same Bull. **51** (1959), 79–86; MR **22** #2674], the author establishes certain generalizations, too lengthy to quote, of results of H. Kestelman [J. London Math. Soc. **22** (1947), 130–136; MR **9**, 274] concerning properties of sets of positive measure in Euclidean spaces under certain linear transformations.

T. A. Botts (Charlottesville, Va.)

8096:

Eisenstadt, B. J.; Lorentz, G. G. Concave functions, rearrangements, and Banach lattices. Michigan Math. J. **7** (1960), 161–170.

Let B be a non-atomic σ-Boolean ring and μ a strictly positive, countably additive measure on B, but

$$\sup_{\mu(f)<+\infty, f\leqq e} \mu(f) = \mu(e).$$

For a countably additive measure ϕ on B with $\phi(e) < +\infty$ for $\mu(e) < +\infty$, putting

$$\phi^*(e) = \sup_{\mu(f)=\mu(e)} \phi(f)$$

we obtain a function ϕ^* on B. The authors attempt a characterization of such a function ϕ^*.

H. Nakano (Kingston, Ont.)

8097:

Volkmann, Bodo. Eine metrische Eigenshaft reeller Zahlkörper. Math. Ann. **141** (1960), 237–238.

The author considers subfields K of the field of all real numbers. Considered as a point set on the line, the Hausdorff-Besicovitch dimension of K is well-defined and $0 \leqq \dim K = \delta \leqq 1$. All known examples of K have $\delta = 0$ or 1, and it is an open question whether or not there exists a K with $0 < \delta < 1$. As a first contribution to the consideration of this problem the author shows that if K is such that $0 < \dim K < 1$, then $\Lambda^\delta(K \cap I)$ is 0 or ∞ for every open interval I, where Λ^δ denotes the Hausdorff measure generated by the function t^δ.

S. J. Taylor (Ithaca, N.Y.)

8098:

Schwartzman, Solomon. Metric transitivity and integer-valued functions. Ann. Inst. Fourier. Grenoble **10** (1960), 297–302.

Let φ be a measurable transformation of a measure space (X, μ) of measure 1 onto itself, let μ be invariant under φ $(\mu(\varphi^{-1}S) = \mu(S)$ for every measurable $S)$, let B denote the set of all measurable integer-valued functions on X, and H_p the set of those non-negative functions in B with values $\leqq p-1$ (by convention, if $f(x) \equiv p-1$, $f \notin H_p$). Let $(p-T)^n$ denote the nth iterate of the operator $(p-T)f(x) = pf(x) - f(\varphi(x))$. The paper is devoted to proving the theorem: The following three conditions are equivalent. (1) φ is metrically transitive. (2) Every $f(x) \in B$ has a unique representation

$$f(x) = k\alpha_0 + (2-T)\alpha_1 + \cdots + (2-T)^n\alpha_n$$

where $\alpha_i \in H_2$, k is a constant and n is the largest integer such that α_n is not identically zero. (3) Every non-negative $f(x) \in B$ has a unique representation $f(x) = \alpha_0 + (p-T)\alpha_1 + \cdots + (p-T)^n\alpha_n$ where $\alpha_i \in H_p$ and n is as in (2).

Y. N. Dowker (London)

8099:

Wright, Fred B. The converse of the Individual Ergodic Theorem. Proc. Amer. Math. Soc. **11** (1960), 415–420.

Let T be a measurable transformation, not necessarily invertible, of a measure space (X, S, m) into itself such that, for the characteristic function f_E of each $E \in S$, the averages $n^{-1}\sum_{j=0}^{n-1} f_E(T^jx)$ converge almost everywhere. It is shown that if T is absolutely continuous there exists a finite invariant measure absolutely continuous with respect to m, while if T is incompressible there exists a finite invariant measure equivalent to m. (T is called absolutely continuous if $m(T^{-1}E) = 0$ whenever $m(E) = 0$, and is called incompressible if $m(T^{-1}E - E) = 0$ whenever $m(E - T^{-1}E) = 0$ for $E \in S$.) This theorem and its proof are extensions of results of the reviewer concerning invertible transformations [cf. Ann. of Math. (2) **54** (1951), 595–608; MR **13**, 543; theorem 6].

Y. N. Dowker (London)

8100:

Wright, Fred B. Structure of measurable transformations. Duke Math. J. **27** (1960), 624–628.

A property of measurable transformations not necessarily invertible is called 'localizable' if for any measurable transformation T, there is an invariant set A such that T restricted to A has the property while T restricted to the complement of A does not have the property.

The author proves that the following properties, among others, are localizable: measure-preserving, existence of an equivalent finite invariant measure, existence of an equivalent σ-finite invariant measure, incompressibility and absolute continuity. (T is called 'incompressible' if $m(T^{-1}E - E) = 0$ whenever $m(E - T^{-1}E) = 0$ and 'absolutely continuous' if $m(T^{-1}E) = 0$ whenever $m(E) = 0$.)

Y. N. Dowker (London)

8101:

Régnier, André. Quelques théorèmes ergodiques ponctuels. C. R. Acad. Sci. Paris **249** (1959), 1077–1078.

Soient E un ensemble, A une σ-algèbre de parties de E et μ une mesure sur A, telle que $\mu(E) < +\infty$. Soit T_s $(s \geqq 0)$ un semigroupe de transformations ponctuelles de E dans E. On dit que A est invariante par les T_s si $X \in A \Rightarrow T_s^{-1}X \in A$ pour chaque s. On dit que A est intégrable pour T_s si (1) A est invariante; (2) $x \in E$ et $X \in A \Rightarrow (T_sx|X)$ est

L-measurable, $(x|X)$ désignant la fonction caractéristique de X; (3) $X \in A$ et $0 \leq t < t' \Rightarrow \int_t^{t'} (Tx_s|X)ds$ est A-mesurable. L'auteur énonce quelques théorèmes ergodiques, par exemple (théorème 3): Dans certaines conditions (par exemple, $X \in A \Rightarrow (T_s x|X)$ est $A \times L$-mesurable) les propositions suivantes sont équivalentes:

$$(1) \qquad X \in A \Rightarrow (1/r) \int_0^r (T_s x|X)\, ds$$

converge μ-presque partout quand $r \to +\infty$;

$$(2) \qquad A \ni X \downarrow \emptyset \Rightarrow \liminf_r (1/r) \int_0^r \mu(T_s^{-1}X)\, ds \downarrow 0;$$

(3) il existe une suite $r_i \to +\infty$ telle que les mesures $(1/r_i)\int_0^{r_i} \mu(T_s^{-1}X)ds$ sont également continues; (4) il existe une suite $r_i \to +\infty$ et une mesure ν sur A telles que $\liminf_{r_i} (1/r_i)\int_0^{r_i} \mu(T_s^{-1}X)ds$ est une fonction de première classe de Baire dans l'algèbre normée des types d'ensembles de A modulo ν. Une σ-algèbre D de parties de nombres ≥ 0 est appelée algèbre densitaire si elle est invariante pour les translations et si $Y \in D \Rightarrow (1/t)\int_0^t (s|Y)ds$ converge quand $t \to 0$. Théorème 4. Soit $T_{s,t}$ ($s \geq 0$, $t \geq 0$) une famille de transformations ponctuelles de E dans E telle que $T_{s',t+s}T_{s,t} = T_{s+s',t}$ et soit D une algèbre densitaire telle que $A \times D$ soit intégrable pour les transformations $T_s(x, t) = (T_{s,t}x, s+t)$ de $E \times R^+$. Si μ est invariante pour toutes les $T_{s,t}$, alors $X \in A$ et $t \geq 0 \Rightarrow (1/r)\int_0^r (T_{s,t}x|X)ds$ converge μ-presque partout quand $r \to +\infty$. Lorsque E est topologique et A est l'algèbre des ensembles boréliens de E, le théorème 4 reste valable si l'on remplace l'existence de D par les conditions suivantes: $T_{s,t}x$ est continue en s pour chaque (t, x) et en (t, x) pour chaque s; il existe $l > 0$ tel que $T_{s,l+t} = T_{s,t}$.

N. Dinculeanu (Bucharest)

8102:

Kemeny, John G. Ergodic theorem for general functions. J. Math. Anal. Appl. 1 (1960), 113–120.

Let (X, S, μ) be a measure space, where the σ-algebra S is generated by a denumerable subalgebra A. A sequence $\nu_1, \nu_2, \cdots$ of measures on X is said to be convergent to a measure ν if $\nu_n(E) \to \nu(E)$ for every set E from A. Let R be an abstract space and M a topological space. To each measure ν on X and each R-valued function f on X is assigned a variance function $V_\nu(f, \cdot)$ on M. It is assumed that the variance function is non-negative, bounded, continuous and takes on a minimum in every closed subset of M. Moreover, for fixed f the mapping $\nu \to V_\nu(f, \cdot)$ is continuous. Let $M_\nu(f)$ be the subset of M on which $V_\nu(f, \cdot)$ takes on its minimum value. For any μ-measure preserving transformation T we put $\mu_{n,x}(E) = (1/n)[c_E(x) + c_E(Tx) + \cdots + c_E(T^{n-1}x)]$, where c_E is the characteristic function of the set E from S. The following generalization of the classical ergodic theorem is proved. For any R-valued function f on X there exists a measure μ_x, depending on x in general, such that $M_{\mu_{n,x}}(f) \to M_{\mu_x}(f)$ a.e., $M_{\mu_{Tx}}(f) = M_{\mu_x}(f)$ a.e. and, if T is ergodic, $M_{\mu_x}(f) = M_\mu(f)$ a.e.

K. Urbanik (Wrocław)

8103:

Abramov, L. M. The entropy of an automorphism of a solenoidal group. Teor. Veroyatnost. i Primenen. 4 (1959), 249–254. (Russian. English summary)

Let X be a solenoid group (a connected, compact, commutative, one-dimensional group). Let T be a continuous automorphism of the topological group X. Then T preserves the Haar measure μ in X and hence is an automorphism of the measure space (X, μ). The character group of X is isomorphic to a subgroup R of the additive group of rational numbers, and the automorphism T^* of R which is adjoint to T is isomorphic to multiplication by a rational number m/n. We assume the fraction m/n to be represented in its lowest terms with n positive. The author proves that the entropy of T is $\log(\max(|m|, n))$.

Y. N. Dowker (London)

FUNCTIONS OF COMPLEX VARIABLES

See also 7982, 8003, 8201, 8279, 8292, 8493, 8495.

8104:

Šneerson, M. S. On an analog of an integral of Cauchy type. Uspehi Mat. Nauk 14 (1959), no. 4 (88), 217–222. (Russian)

Soit D un domaine à trois dimensions, p une fonction scalaire et $a = (a_x, a_y, a_z)$ une fonction vectorielle définies sur D. On pose:

$$(p; a) = \begin{pmatrix} p & a_z & -a_y \\ -a_z & p & a_x \\ a_y & -a_x & p \end{pmatrix}.$$

On dit que la matrice $(p; a)$ est monogène de premier [resp. de second] ordre si p et a ont des dérivées continues de premier [resp. de second] ordre dans D et si grad p + rot $a = 0$ [resp. grad p − rot $a = 0$] dans D. Si, en outre, div $a = 0$, on dit que $(p; a)$ est fortement monogène de premier [resp. second] ordre. Pour $a = (a_x, a_y, a_z)$ et $b = (b_x, b_y, b_z)$ on pose

$$b \cdot a = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}.$$

On démontre le théorème suivant (analogue au théorème de Cauchy pour les fonctions complexes): Soit S une surface fermée telle que Int $S \subset D$. Si la matrice $(p; a)$ est fortement monogène de premier ordre, alors

$$\iint_S \{(p; a)(0; n) - a \cdot n\}\, dS = 0,$$

où n est la normale extérieure de S. On déduit aussi des analogues de la formule de Pompeiu et de la formule intégrale de Cauchy et on définit une intégrale de type de Cauchy. L'auteur étudie ensuite les fonctions analytiques de matrice.

N. Dinculeanu (Bucharest)

8105:

Şneerson, M. S. On an analogue to an integral of Cauchy type. Acad. R. P. Romîne. An. Romîno-Soviet Ser. Mat.-Fiz. (3) 14 (1960), no. 1 (32), 58–65. (Romanian)

Translation of #8104.

8106:

Merkes, E. P. Bounded J-fractions and univalence. Michigan Math. J. 6 (1959), 395–400.

If $M \geqq 0$ and $N > 0$, $J(M, N)$ denotes the set to which the function f from $|z| > \frac{1}{2}(2N+M)$ belongs only in case there is a complex number sequence a and a complex number sequence b such that $|a_p| \leqq \frac{1}{2}N$, $|b_p| \leqq \frac{1}{2}M$ $(p=1, 2, \cdots)$, and

$$f(z) = \frac{1}{b_1+z-} \frac{a_1^2}{b_2+z-} \frac{a_2^2}{b_3+z-} \cdots.$$

Thale [Proc. Amer. Math. Soc. **7** (1956), 232–244; MR **17**, 1063] showed that each function in $J(M, N)$ is univalent for $|z| > \frac{1}{2}(3 \cdot 2^{1/2}N + 2M)$. No domain of univalence for $J(M, N)$ properly includes this circular domain.

On writing the bounded J-fraction for f in $J(M, N)$ in the form

$$\frac{1}{b_1+z}\left\{\frac{1}{1+} \frac{-a_1^2/(b_1+z)(b_2+z)}{1} + \frac{-a_2^2/(b_2+z)(b_3+z)}{1} + \cdots\right\}$$

and noting that $|a_p^2/(b_p+z)(b_{p+1}+z)| < \frac{1}{4}$ if $|z| > \frac{1}{2}(2N+M)$, the author applies an inequality of Paydon and Wall [Duke Math. J. **9** (1942), 360–372; MR **3**, 297; Theorem 9.1, p. 365] to show that the map of $|z| > \frac{1}{2}(3 \cdot 2^{1/2}N + 2M)$ by any f in $J(M, N)$ includes the domain $|w| < 3/2(2^{1/2}N+M)$ and is included by the domain $|w| < 3 \cdot 2^{1/2}/2N$. The result is sharp. He obtains a lower bound for the radius of starlikeness of $J(M, N)$ and a lower bound for the radius of convexity. *H. S. Wall* (Austin, Tex.)

8107:

MacNerney, J. S. Investigation concerning positive definite continued fractions. Duke Math. J. **26** (1959), 663–677.

This paper contains a proper extension of the theory of positive definite continued fractions. A positive definite continued fraction is generated by an infinite sequence of linear fractional transformations of the form

$$(1) \quad t_p(U) = A_{p-1}^2/[Z_p - B_p - U] \quad (p = 1, 2, \cdots),$$

where a certain J-matrix is non-negative definite or, what is equivalent, there exists a number sequence g such that the complex number sequences A and B satisfy

$$(2) \quad |A_p^2| - \operatorname{Re} A_p^2 \leqq 2(1-g_{p-1})g_p b_p b_{p+1}, \; 0 \leqq g_{p-1} \leqq 1, \; 0 \leqq b_p = \operatorname{Im}(-B_p), \quad (p = 1, 2, \cdots)$$

[cf. Wall, *Analytic theory of continued fractions*, Van Nostrand, New York, 1948; MR **10**, 32; esp. Chapter IV]. In the extension, the system (1) is replaced by

$$(3) \quad t_p(U) = A_{p-1}[Z_p - B_p - U]^{-1}C_{p-1}, \quad (p = 1, 2, \cdots),$$

where A_{p-1}, C_{p-1}, B_p, Z_p and U are continuous linear transformations in a complete inner product space S, e.g., $n \times n$ matrices. If $((x, y))$ is the inner product and x^* the adjoint of x in S, the condition for positive definiteness is

$$\sum_{p=1}^{n+1} ((b_p x_p, x_p)) - 2\operatorname{Re} \sum_{p=1}^{n} ((a_p x_{p+1}, x_p)) \geqq 0$$

for each sequence x with values in S, where $b_p = \operatorname{Im}(-B_p)$ and $a_p = i[C_p^* - A_p]/2$ $(p=1, 2, \cdots)$. An equivalent formulation in terms of "chain sequences", analogous to (2), is found, the g_p now being Hermitian continuous linear transformations in S; the notions "maximal parameters" and "minimal parameters" of the chain sequence are found for the extended theory.

A "nest of circles" for the extended theory is obtained and those analytic functions having their values in all the circles for $\operatorname{Im} z > 0$ are characterized in terms of Stieltjes integrals. *H. S. Wall* (Austin, Tex.)

8108:

Corrádi, Keresztély. Über Konvergenz-Eigenschaften von Potenzreihen. Mat. Lapok **10** (1959), 136–141. (Hungarian. Russian and German summaries)

It is known that a power series may converge uniformly on the unit circle, but not absolutely, or again may converge everywhere on the unit circle and represent an unbounded function inside the circle. Here very simple examples are constructed of these phenomena, namely, $\sum_n P_n(z)$, where in the first case

$$P_n(z) = \frac{z^{2^{n+1}}}{n2^{n+1}} \prod_{k=0}^{n} \{1+(-1)^k z^{2^k}\},$$

and in the second case

$$P_n(z) = \frac{z^{n^2}}{2^n\sqrt{\log n}}(1-z)^n.$$

For references see P. Turán, Bull. Amer. Math. Soc. **54** (1948), 932–936 [MR **10**, 241]. *F. V. Atkinson* (Toronto)

8109:

Weiss, Mary. Concerning a theorem of Paley on lacunary power series. Acta Math. **102** (1959), 225–238.

1. Soit $\sum c_k z^{n_k}$, avec $n_{k+1}/n_k > q > 1$, $c_k \to 0$, $\sum |c_k| = \infty$, et soit ζ un nombre complexe quelconque. Il existe un point ξ, $|\xi| = 1$, tel que $\sum c_k \xi^{n_k} = \zeta$. 2. Il existe deux constantes A_q et A_q' possédant les propriétés suivantes: Si le polynome $T(\theta) = \sum (a_k \cos n_k\theta + b_k \sin n_k\theta)$ satisfait l'inégalité $T(\theta) \leqq M$ dans un intervalle $a \leqq \theta \leqq b$, avec $b-a \geqq A_q/n$, on a aussi $\sum (|a_k| + |b_k|) \leqq A_q'M$. Les deux théorèmes ont été énoncés, mais non démontrés, par Paley. L'auteur les démontre. *S. Mandelbrojt* (Paris)

8110:

Motzkin, T. S. Power series with gaps. Proc. Amer. Math. Soc. **11** (1960), 875.

An earlier theorem of the author [Math. Ann. **109** (1933), 95–100] asserts that if $a_n = 0$ whenever n belongs to one of k residue classes ($k=2$ or $k=3$) satisfying certain restrictions, then $\sum a_n z^n$ has at least $k+1$ singularities on its circle of convergence. The author shows that the theorem fails for $k=4$ and for $k \geqq 6$.
G. Piranian (Ann Arbor, Mich.)

8111:

Kimura, Tosihusa. Sur la propriété d'Iversen et l'équation différentielle ordinaire du second ordre. Comment. Math. Univ. St. Paul **8** (1960), 63–70.

An analytic function $f(z)$ is said to have the Iversen property in a domain D if it satisfies the following conditions: (a) $f(z)$ has at least one function element $P(z-a)$ with $a \in D$; (b) given a function element $P(z-a)$ and a path L in D connecting a with a point b, $P(z-a)$ can be continued analytically along a path which is arbitrarily close to L and ends in an arbitrarily small neighborhood of b. The following theorem is proved: Let $P(z, y, y')$ and

$Q(z, y, y')$ be polynomials in y and y'—of degrees p and q, respectively, in y'—whose coefficients are analytic in a domain D. If $p>q+2$, every integral of the differential equation $y''=P(z, y, y')/Q(z, y, y')$ has the Iversen property in D. *Z. Nehari* (Pittsburgh, Pa.)

8112:

Bergström, Harald. An approximation of the analytic function mapping a given domain inside or outside the unit circle. Les mathématiques de l'ingénieur, pp. 193–198. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

The function which conformally maps a given domain on to the interior or exterior of a unit circle is determined by a method of successive approximations. Three examples which show the method to be a practical one are given.
W. D. Collins (Newcastle-upon-Tyne)

8113:

Andrianov, S. N. Conformal mapping of many-sheeted regions on the circle. Issledovaniya po sovremennym problemam teorii funkcil kompleksnogo peremennogo, pp. 332–335. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let $\Gamma: w=w(\sigma)$ $(0\leqq\sigma\leqq l)$ be a sufficiently smooth closed curve in the complex w-plane, σ being a parameter denoting arc-length of Γ. The work is concerned with the problem of finding functions $f(z)$ and $\sigma(\gamma)$ satisfying the conditions (1) $f(z)$ is regular inside the unit circle E in the complex z-plane and continuous in $\bar{E}$ and (2) $\sigma(\gamma)$ $(\sigma(0)=0,\ \sigma(2\pi)=l)$ is strictly increasing and absolutely continuous, such that $w(\sigma(\gamma))=f(e^{i\gamma})$ $(0\leqq\gamma\leqq 2\pi)$ at every point $z=e^{i\gamma}$ on the circumference $|z|=1$.

The author reduces this problem to that of solving two singular integral equations, and remarks that the method of iteration is available to solve them. The proofs of the results obtained are briefly described. *T. Kubo* (Kyoto)

8114:

Husemoller, Dale H. A generalization of Hurwitz's theorem. Proc. Glasgow Math. Assoc. 4, 101–102 (1960).

Let X and Y be two Riemann surfaces, and let $A_c(X, Y)$ be the set of all analytic maps of X into Y. Topologize $A_c(X, Y)$ with the compact-open topology. Let $E_n(x, y)$ be the subset of $A_c(X, Y)$ consisting of those f for which $f(x)=y$ with multiplicity n (multiplicity 0 means $f(x)\neq y$), and let $F_n(y)$ be the subset of $A_c(X, Y)$ consisting of those f for which y is assumed as a value exactly n times counting multiplicities. Let $\bar{E}_n(x, y)$ and $\bar{F}_n(y)$ be the same subsets with the addition of the constant functions. Then the author proves the following theorem: Let X and Y be two Riemann surfaces and let $x\in X$ and $y\in Y$; then $E_n(x, y)$ is open, $F_n(y)$ is open, and $\bigcup_{r\leq n}\bar{F}_r(y)$ is closed in the topology on $A_c(X, Y)$.
H. L. Royden (Stanford, Calif.)

8115:

Rabinovič, Yu. L. Entire functions representable as Laplace integrals. II. Issledovaniya po sovremennym problemam teorii funkcil kompleksnogo peremennogo, pp. 186–194. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

[Part I appeared in Moskov. Gos. Univ. Uč. Zap. **181**. Mat. 8 (1956), 199–221; MR **19**, 403.] Let L_α' denote the class of functions $f(\zeta)$ regular in the angle $-\pi/2-\pi/\alpha<\arg\zeta<-\pi/2$, $\alpha>1$, with $\liminf\{\log\log[1/|f(\zeta)|]\}/\log|\zeta|=\lambda>1$ in each interior angle, and $|f(\zeta)|=O(|\zeta|^{\beta-1})$, $\beta>0$, in each neighborhood of 0 in each interior angle. Let L_α'' denote the class of entire functions F of finite order $\nu\geqq\alpha/(\alpha-1)$ with $|F(z)|=O(|z|^{-\beta})$ in each angle $\delta\leqq\arg z\leqq\pi+\pi/\alpha-\delta$. Theorem: F is the Laplace transform of a function in L_α' if and only if $F\in L_\alpha''$ (and then $\nu^{-1}+\lambda^{-1}=1$). *R. P. Boas, Jr.* (Evanston, Ill.)

8116:

Shah, S. M. Note on an entire function of infinite order. Math. Student **27** (1959), 9–12.

In a joint paper [Proc. Royal Soc. Edinburgh Sect. A **64** (A) (1954), 80–89; MR **16**, 122] the author and S. K. Singh have proved this result: Let (i) $\theta(x)$ be >0 and non-decreasing for $x\geqq x_0$, $\theta(x)\to\infty$ $(x\to\infty)$; (ii) $I(x)=\int_{x_0}^x(t\theta(t))^{-1}dt$, $I(x)\to\infty$ $(x\to\infty)$; (iii) $x\theta'(x)/\theta(x)\leqq c<1$ $(x>x_0)$. Then, if N is an integer such that $I(N)\geqq 1$, $f(z)=\sum_N^\infty(z/I(n))^n$ is an entire function of infinite order such that $\log\mu(r)\theta(\log\mu(r))/\nu(r)\to 0$ as $r\to\infty$. (iv) If in addition θ is such that $I(n^p)-I(n)>2/\theta(n+1)$ for a fixed integer p and large n, then

$$\log M(r)\theta(\log M(r))/\nu(r)\to 0\quad(r\to\infty);$$

where $M(r)$ is the maximum modulus of $f(z)$, $\nu(r)$ the rank of the maximum term in the power series of f and $\mu(r)$ the maximum term.

It was shown by J. Clunie [Quart. J. Math. Oxford Ser. (2) **7** (1956), 175–182; MR **20** #2444] that condition (iv) can be dropped. The author arrives at the same result by another method, and he shows by a counter-example (Gegenbeispiel) that (iv) is not implied by (i)–(iii).
H. Kober (Birmingham)

8117:

Hiong, King-lai. Sur la théorie des défauts relative aux fonctions méromorphes dans le cercle unité. Sci. Sinica **9** (1960), 575–603.

Methods of the Nevanlinna value distribution theory are applied to estimate the growth of a function $f(x)$, meromorphic in $|x|<R\leqq\infty$ and with a sum of deficiencies exceeding a certain given number. The theorem of Landau is generalized as follows: If $f(x)=c_0+c_1x+\cdots$, $c_0, c_1\neq 0$, is holomorphic in $|x|<R$, z_ν, $\nu=1,\cdots,q$, $q>1$, are deficient values for $f(x)$ with a total deficiency $l>1$, and $f(0)\neq z_\nu$, then $|c_1|R$ is bounded by a number depending on l and on the values z_ν. Most of the other results are too complicated to be restated here. *O. Lehto* (Helsinki)

8118:

Bagemihl, F.; Piranian, G.; Young, G. S. Intersections of cluster sets. Bul. Inst. Politehn. Iaşi (N.S.) **5 (9)** (1959), no. 3–4, 29–34. (Russian and Romanian summaries)

Let f be a complex-valued function defined in $D:|z|<1$. Relative to each function f, the authors investigate the existence of triplets of Jordan arcs A_1, A_2, A_3 in D which have one common endpoint $e^{i\theta}$ and for which the intersection $C(f, A_1)\cap C(f, A_2)\cap C(f, A_3)$ is empty; here

$C(f, A)$ denotes the cluster set of f at $e^{i\theta}$ along A. If the intersection of the three cluster sets is empty, for some triplet of arcs terminating at $e^{i\theta}$, f is said to have the three-arc property at $e^{i\theta}$; if moreover the three arcs can be taken to be rectilinear segments, f is said to have the three-segment property at $e^{i\theta}$. The authors prove: There exists a function in D which has the three-segment property at each point $e^{i\theta}$. There exists a continuous function in D which has the three-segment property at each point of a perfect set on $|z|=1$. The elliptic modular function has the three-arc property at every point $e^{i\theta}$. There exists a Blaschke product which has the three-arc property at each point of a perfect set on $|z|=1$. Furthermore, two theorems on many-segment properties are proved. Some related open questions are raised.

K. Noshiro (Nagoya)

8119:

Collingwood, E. F. Cluster sets of arbitrary functions. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1236–1242.

Let Γ be the unit circle and D be the open unit disk in the complex plane, and suppose that $f(z)$ is an arbitrary single-valued (real or complex) function in D. Then $C_{Br}(f, e^{i\theta})=C_{Bl}(f, e^{i\theta})=C(f, e^{i\theta})$ except perhaps for an ennumerable set of points $e^{i\theta}\in\Gamma$. With the possible exception of a set of points $e^{i\theta}$ of first category in Γ, (i) $C_\Delta(f, e^{i\theta})=C(f, e^{i\theta})$, where Δ is any Stolz angle at $e^{i\theta}$; (ii) $C_{\Lambda(\theta)}(f, e^{i\theta})=C(f, e^{i\theta})$, where $\Lambda(\theta)$ is the domain obtained by rotation about the origin, through the angle θ, of a subdomain $\Lambda(0)$ of D that is arbitrarily narrow near the point 1 lying on its frontier; (iii) $C_{\lambda'_\theta}(f, e^{i\theta})=C(f, e^{i\theta})$, where λ_θ is the Jordan arc obtained by rotation about the origin, through the angle θ, of a Jordan arc λ_0 in D that terminates in the point 1, and λ_θ' is, in a certain sense, arbitrarily close to λ_θ.

F. Bagemihl (Ann Arbor, Mich.)

8120:

Reich, Edgar; Warschawski, S. E. On canonical conformal maps of regions of arbitrary connectivity. Pacific J. Math. **10** (1960), 965–985.

Let Ω be a bounded domain which contains the origin, and let $\mathscr{F}=\mathscr{F}(\Omega)$ denote the family of analytic functions $f(z)$ univalent in Ω and satisfying $f(0)=0$, $\arg f'(0)=0$ and $|f(z)|\leqq 1$ in Ω. The authors show that the extremal problem of maximizing $f'(0)$ within $\mathscr{F}$ is solved by a function which maps Ω onto a circular slit disk. The proof is based on a principal lemma which may be regarded as a generalization of a classical inequality of Rengel. A subclass $\mathscr{F}^*=\mathscr{F}^*(\Omega)$ of $\mathscr{F}$ is defined by a restrictive requirement that the outer boundary component of Ω is carried by f onto the unit circumference. Considering then the corresponding maximizing problem for $\mathscr{F}^*$, the authors prove the existence and mapping behavior of the solution together with its uniqueness. Some properties of extremal functions of $\mathscr{F}$ and $\mathscr{F}^*$ are derived. A circular slit disk Σ is called extremal if the extremal function of $\mathscr{F}^*(\Sigma)$ is the identity. It is shown that the extremality of Σ can be characterized in terms of extremal length.

Suppose further that the outer boundary component of Ω is free. Then, similar discussions are made for mapping onto a radial slit disk by replacing the maximizing problem for $\mathscr{F}^*(\Omega)$ by the corresponding minimizing one.

Y. Komatu (Tokyo)

8121:

Maksimov, Yu. D. Extension of the structural formula for convex univalent functions to a multiply connected circular region. Dokl. Akad. Nauk SSSR **136** (1961), 284–287 (Russian); translated as Soviet Math. Dokl. **2**, 55–58.

The author generalizes well known structure formulas for convex univalent functions regular in the circle to regions bounded by a system of circles: $|z-a_0|\leqq r_0$ excluding the circles $|z-a_k|\leqq r_k$ $(k=1, 2, \cdots, n)$, where $|a_k-a_j|>r_k+r_j$ $(k=1, 2, \cdots, n;\ k\neq j)$, and $|a_k-a_0|<r_0$. The formulas are lengthy. No proofs are given.

W. C. Royster (Lexington, Ky.)

8122:

Biernacki, M. Sur l'intégrale des fonctions univalentes. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 29–34. (Russian summary, unbound insert)

The author uses variational methods to prove the following. If $f(z)=a_1z+\cdots$ is regular and univalent in $|z|<1$ then the function $g(z)=\int_0^z f(z)/z\,dz$ is also univalent in this circle. There is one point at which the reviewer is unable to follow the proof.

A. W. Goodman (Lexington, Ky.)

8123:

Turán, P. On the infinite product representation of functions regular and nonvanishing in the unit circle. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 481–486. (Russian summary, unbound insert)

Let $F(z)$ be regular in $|z|<1$, continuous in $|z|\leqq 1$, with $F(0)=1$. Then $F(z)$ can be represented as a series $\sum_1^\infty d_n \log f_n(z)$, where the d_n are real, and the $f_n(z)$ have the form $F_\nu\{z \exp(2\pi i(2k-1)2^{-\nu})\}$, where

$$F_\nu(z) = (1-z)^2(1-z\exp(2\pi i/2^\nu))^{-1}(1-z\exp(-2\pi i/2^\nu))^{-1}.$$

The series converges uniformly for $|z|\leqq 1-\varepsilon$, while its imaginary part converges uniformly for $|z|\leqq 1$. For the proof, Im $F(z)$ is expanded by means of a modified Haar orthogonal system on $|z|=1$.

F. V. Atkinson (Toronto)

8124:

Lebedev, N. A.; Sogomonova, G. A. A method of obtaining a certain kind of estimate for functions regular in the circle. Vestnik Leningrad. Univ. **14** (1959), no. 13, 15–19; addendum **15** (1960), no. 13, 152. (Russian. English summary)

Let $f(z)$ and $F(z)$ be regular for $|z|<1$. If $F(z)$ is schlicht, and if there exists a function $\omega(z)$ regular for $|z|<1$, such that $|\omega(z)|<1$ and $f(z)=F(\omega(z))$, then f is called subordinate to F. Generalizing a theorem of Rogosinski [Königsberg. Gelehrt. Gesellschaft. Naturwiss. Kl. **8** (1931), 1–31] the authors utilize the concept of subordination in the following way. Suppose (i) $F_\lambda(z)$ is continuous in λ for each z, $\lambda_1\leqq\lambda\leqq\lambda_2$; (ii) the images of $\{|z|<1\}$ under F_λ increase with λ; (iii) the class $N(a_0, a_1, \cdots, a_m)$ of functions $f(z)=\sum_0^\infty a_\nu z^\nu$ subordinate to $F_{\lambda_2}(z)$, and with $a_0, a_1, \cdots, a_m$ (not all zero) given, is non-empty. Theorem: There exists a number λ_0, $\lambda_1\leqq\lambda_0\leqq\lambda_2$ (the root of a certain determinantal equation), such that at most one function $f_0(z)\in N$ is subordinate to $F_{\lambda_0}(z)$. As applications some best inequalities are obtained for the classes $M(a_0, \cdots, a_m)$ of functions regular for $|z|<1$, with given initial coefficients $a_0, \cdots, a_m$, and the subclass $M^0\subset M$ for which

$f(z)$ has no zeroes. For example, the authors obtain a best inequality of the type $\sup_{|z|<1} |\mathscr{R}\{e^{-i\theta} \log f(z)\} + c| \geqq \lambda_0$ for given real θ, c, $f \in M^0(a_0, 0, \cdots, 0, a_m)$. *E. Reich* (Aarhus)

8125:

Zamorski, J. The estimation of the third coefficient of the starlike function with a pole. Ann. Polon. Math. **8** (1960), 185–191.

Let

$$f(z) = z^{-1} + a_0 + a_1 z + \cdots$$

be univalent in $|z| < 1$ and map $|z| < 1$ onto the complement of a point set starlike with respect to the origin. If $a_0 = 0$, it is known that the coefficients a_n $(n \geqq 1)$ are subject to the sharp inequality (*) $|a_n| \leqq 2(n+1)^{-1}$. (For $1 < n \leqq 6$, this was shown by the reviewer and E. Netanyahu [Proc. Amer. Math. Soc. **8** (1957), 15–23; MR **18**, 648] and the general result was obtained by J. Clunie [J. London Math. Soc. **34** (1959), 215–216; MR **21** #5737].) The author shows that the inequality $|a_3| \leqq \frac{1}{2}$ remains true for general values of a_0. {Reviewer's note: Clunie's proof of (*) (which apparently was not known to the author) can be easily modified so as to yield the inequality (*) without the assumption $a_0 = 0$.} *Z. Nehari* (Pittsburgh, Pa.)

8126a:

Jenkins, James A. On certain coefficients of univalent functions. Analytic functions, pp. 159–194. Princeton Univ. Press, Princeton, N.J., 1960.

8126b:

Jenkins, James A. An extension of the general coefficient theorem. Trans. Amer. Math. Soc. **95** (1960), 387–407.

In numerous preceding papers the author has developed a general coefficient theorem (G.C.T.) and applied it systematically to the theory of univalent functions and to mappings of finite Riemann surfaces [see *Univalent functions and conformal mapping*, Berlin, 1958; MR **20** #3288]. The G.C.T. deals with a finite oriented Riemann surface R, a quadratic differential Q on it, and subdomains $\Delta \subset R$ which are bounded by curves with the differential equation $Qdz^2 > 0$. Let $f(z)$ be meromorphic in Δ, map it conformally into R and satisfy various additional conditions. Then, inequalities are asserted which involve the power series coefficients of $f(z)$ and $Q(z)$ at each pole of $Q(z)$. If $Q(z)$ has near $z = \infty$ the development $Q(z) = \alpha z^{m-4} +$ lower powers (i.e., pole of order m) with $m \geqq 3$, the admissible functions have to satisfy the gap condition $f(z) = z + az^{-(m-3)} +$ lower powers. The G.C.T. is closely related to Teichmüller's work on univalent functions [S.-B. Preuss. Akad. Wiss. **1938**, 363–375]. In the present papers an important generalization of the G.C.T. is established and illustrated by many applications. The requirement on admissible $f(z)$ is relaxed to the gap condition $f(z) = z + az^{-k} + \cdots$ with $k \geqq \frac{1}{2}m - 2$. In the first paper, special cases are proved from the elements. A typical result is the following. Let Δ be a simply-connected domain in the complex z-plane which contains the point $z = \infty$ but not $z = 0$ and which is bounded by a curve with the differential equation $\alpha dz^2(1 + \beta/z) > 0$. If $f(z) = z + a_0 + a_1/z + \cdots$ is univalent in Δ and $f(z) \neq 0$, then $\mathrm{Re}\{\alpha(a_1 + \beta a_0)\} \leqq 0$. This result is used to derive many inequalities for the first two coefficients of normalized functions univalent in the unit circle. Considering quadratic differentials with higher poles at infinity, the author derives further inequalities for the first three coefficients of normalized univalent functions, among them $|c_3| \leqq \frac{1}{2} + e^{-6}$ for the third coefficient of a univalent function in the exterior of the unit circle [see Garabedian and Schiffer, Ann. of Math. (2) **61** (1955), 116–136; MR **16**, 579]. In the second paper the general result is proved and discussed. As illustrative example, the following problem is solved. Let $f(z) = z + a_0 + a_1/z + \cdots$ be univalent in $|z| > 1$ and map onto a domain whose complement contains the origin and has the inner radius r with respect to it. Give precise bounds for $|a_1|$ in terms of r.

M. Schiffer (Stanford, Calif.)

8127:

Pykhteev, G. N. On two representations of functions, analytic in the upper half-plane. Problems of continuum mechanics (Muskhelishvili anniversary volume), pp. 350–358. SIAM, Philadelphia, Pa., 1961.

The author discusses an expansion theorem of a function $\phi(z)$ analytic in the upper half-plane which satisfies on the real axis the conditions (i) $\mathrm{Re}\, \phi(x) = 0$ for $|x| \geqq 1$, (ii) $\phi(x) = f_1(x) + if_2(x)$ for $-1 \leqq x \leqq 1$, and the value $\phi(\infty)$ is given. It is shown that there exist expansions involving functions which are infinite series developments of Chebyschev polynomials of the first and second kind. The derivation is function-theoretic in character and the conditions on the f's, which are too detailed to discuss here, are given. Such representations are said to be useful in planar problems of hydrodynamics, aerodynamics and elasticity.

A. E. Heins (Ann Arbor, Mich.)

8128:

Hayman, W. Interpolation by bounded functions. Ann. Inst. Fourier. Grenoble **8** (1958), 277–290, xii. (French summary)

The author considers the following problem, posed in a lecture by R. C. Buck: which sequences $\{z_n\}$ in the unit circle have the property that an arbitrary bounded sequence $\{w_n\}$ can be interpolated at these points by means of a bounded analytic function. Such a sequence $\{z_n\}$ is called a universal interpolating sequence (u.i.s.). Let $r_{mn} = |z_m - z_n| / |1 - \bar{z}_m z_n|$, and

$$P_n(\lambda) = \prod_{m \neq n} [1 - (1 - r_{mn})^\lambda].$$

The author shows that the condition (I) $P_n(1) \geqq c > 0$ is necessary for $\{z_n\}$ to be a u.i.s., and that the condition (II) $P_n(\lambda) \geqq c > 0$ for some $\lambda < 1$ is sufficient. Condition (I) is equivalent to the assertion that the special bounded sequences $\{\delta_{nk}\}$ $(n = 1, 2, \cdots)$ can be interpolated by uniformly bounded functions $(f_n(z_k) = \delta_{nk}$, $|f_n(z)| \leqq 1/c$ for all n, k and $z)$. The necessity of (I) can then be deduced from the closed graph theorem; however, the author gives a direct proof. He shows in fact that if arbitrary sequences of zeros and ones can be interpolated, then condition (I) holds.

In proving the sufficiency the author constructs functions $h_n(z)$ such that (i) $h_n(z_k) = \delta_{nk}$, and (ii) $\sum |h_n(z)| \leqq c_1$ for all z. The function $\sum w_n h_n(z)$ solves the interpolation problem.

The author shows that if $\{z_n\}$ approach the boundary

exponentially, that is, if $\lim\sup(1-|z_{n+1}|)/(1-|z_n|)<1$, then $\{z_n\}$ is a u.i.s. If the points z_n all lie on one radius then this condition is also necessary.

{See the following reviews of independent papers on the same problem by Carleson [#8129] and Newman [#8130]. Carleson showed that (I) is necessary and sufficient for $\{z_n\}$ to be a u.i.s. Another proof of this result has been obtained recently by H. S. Shapiro and the reviewer.}
A. L. Shields (New York)

8129:

Carleson, Lennart. An interpolation problem for bounded analytic functions. Amer. J. Math. **80** (1958), 921–930.

Let B be the algebra of all functions analytic and bounded on the open disk $\Delta:|z|<1$. Let $z_n\in\Delta$, with $\lim|z_n|=1$. It is well-known that the condition $\sum(1-|z_n|)<\infty$ is necessary and sufficient for there to exist a function $\pi\in B$, with $\pi(z)=0$ exactly when $z=z_1, z_2, \cdots$. At the Princeton Analysis Conference, the reviewer made the conjecture that if $|z_n|\to 1$ rapidly enough, then for any bounded sequence $\{w_n\}$, there would exist a function $f\in B$ such that $f(z_n)=w_n$, $n=1, 2, \cdots$. More generally, one seeks necessary and sufficient conditions on $\{z_n\}$ for it to have this universal interpolation property. Partial solutions were obtained by the reviewer, Gleason, Newman; Hayman [see review #8128] found that the condition

$$(*)\qquad \prod_{n\neq k}|z_n-z_k|/|1-z_n\bar z_k| \geq \delta>0$$

for $k=1, 2, \cdots$ and some constant δ, was necessary, and was able to show that a slightly stronger condition was sufficient. In the present paper, the author gives an indirect ingenious proof that (*) is also sufficient. Let F_n be the class of functions in B that obey the condition $f(z_k)=w_k$, $k=1, 2, \cdots, n$. Let $M_n(w)=\inf_{f\in F_n}\sup_{z\in\Delta}|f(z)|$. Then, it must be shown that $\{M_n(w)\}$ is bounded for all sequences $\{w_n\}$ with $|w_n|\leq 1$. The author first shows that $M_n(w)$ can be given alternatively as

$$\sup_G \Big|\sum_1^n G(z_k)w_k/\pi_n'(z_k)\Big|,$$

where G ranges over the functions in H^1 with $\|G\|_1\leq 1$ and π_n is the Blaschke product with zeros at $z_1, z_2, \cdots, z_n$. When (*) holds, the problem reduces to showing that $\sum(1-|z_k|^2)|G(z_k)|=O(1)$ for all G.

In the last section, the author returns to the problem which gave rise to the conjecture, the algebraic structure of B. He shows that if $\{z_n\}$ is a universal interpolating sequence, and π is its Blaschke product, and $f\in B$ with $|f(z)|+|\pi(z)|\geq\delta>0$ for all $z\in\Delta$, then the ideal generated in B by f and π is all of B. {A simple condition that implies (*), and hence universality for $\{z_n\}$, is $1-|z_{n+1}|\leq \rho(1-|z_n|)$, $n=1, 2, \cdots$, where $\rho<1$.}
R. C. Buck (Princeton, N.J.)

8130:

Newman, D. J. Interpolation in H^∞. Trans. Amer. Math. Soc. **92** (1959), 501–507.

What condition on the distribution of the complex numbers a_n ($|a_n|<1$) is necessary and sufficient for the following interpolation property to hold: Given any bounded complex sequence $\{w_n\}$ there exists a function $F(z)$ analytic and bounded in $|z|<1$ for which $F(a_n)=w_n$? Using an ingenious and simple Banach space argument [reminiscent of, say, the one used in Helson, Studia Math. **14** (1954), 209–213; MR **16**, 817], the author obtains the following answer to this question. Let

$$B(z)=\prod_n(a_n-z)(a_n-|a_n|^2z)^{-1},$$

the Blaschke product formed from the a_n. The interpolation property holds if and only if: (1) $B(z)$ converges in $|z|<1$, and, for all n, $(1-|a_n|^2)|B'(a_n)|\geq \text{const}>0$; (2) $\sum_n(1-|a_n|^2)|G(a_n)|<\infty$ for $G(z)\in H_1$ (that is, for $G(z)=\sum c_nz^n$ regular in $|z|<1$ with $\sum c_ne^{in\theta}$ the Fourier series of a function in $L_1(-\pi,\pi)$).

The author applies this to obtain various results, of which we mention the following: Let $0<a_n<1$, the a_n being arranged in increasing order. A necessary and sufficient condition for the interpolation property to hold is the existence of a $k<1$ with $1-a_n\leq k(1-a_{n-1})$ for all n.

{It is remarkable that condition (1) alone is sufficient, as well as necessary, for the interpolation property to hold. This most elegant result is contained in a paper by L. Carleson [see preceding review] published before Newman's, although both papers were submitted for publication at about the same time. Using more classical methods, Carleson first obtains what is essentially the above solution. He then deduces, by means of a curious and delicate "elementary" argument that condition (1) implies condition (2). It would be very interesting to find a proof of this implication which is perhaps less elementary, but simpler, than that of Carleson.} *P. Koosis* (New York)

8131:

Baillette, Aimée. Approximation de fonctions par des sommes d'exponentielles. C. R. Acad. Sci. Paris **249** (1959), 2470–2471.

Soient $\Lambda=\{\lambda_n\}$ une suite réelle positive, $\mathfrak{H}(\Omega)$ l'espace vectoriel des fonctions holomorphes dans un ouvert Ω (topologie de la convergence compacte), $\mathfrak{H}_\Lambda(\Omega)$ le sous-espace fermé engendré par $\{e^{-\lambda_n z}\}$. En généralisant quelques résultats de J.-P. Kahane [*Mean periodic functions*, Tata Institute of Fundamental Research, Bombay, 1959], l'auteur cherche, à partir de Λ, Ω, les domaines Δ tels que toute $f\in\mathfrak{H}_\Lambda(\Omega)$ puisse être prolongée dans Δ avec $\sup_{z\in\Delta}|f(z)|\leq A\sup_{z\in\Omega}|f(z)|$, A ne dépendant que de Λ, Ω et Δ. *S. Mandelbrojt* (Paris)

8132:

Manihin, V. M. Some questions of weighted approximations of functions. Uč. Zap. Borisoglebsk. Gos. Ped. Inst. **1958**, no. 5, 133–144. (Russian)

Soient (r_n) une suite de fonctions positives continues définies sur $[-1,1]$, telle que $\lim_{n\to\infty}(r_n(x))^{1/n}=|x|^a$, $a\geq 0$, et f une fonction numérique continue sur $[-1,1]$. Posons $E_n=\inf_{P_n}\sup_{-1\leq x\leq 1}|(f(x)-P_n(x))/r_n(x)|$ où P_n désigne un polynom de degré $\leq n$. L'auteur démontre les résultats suivants: (1) Si $a<1$ et si $\lim\sup(E_n)^{1/n}=1/\Lambda$, $\Lambda>1$, alors f est analytique dans le domain $D:|z|^a|z+(z^2-1)^{1/2}|^{1-a}<\Lambda$; (2) si $r_n(x)=|x|^{a_n}$, $a_n/n\to a<1$ et si f est analytique dans D, alors $\lim\sup(E_n)^{1/n}=1/\Lambda$; (3) si $a>1$ et si $\lim\sup(E_n)^{1/n}=C$, alors f est entière. L'auteur étude aussi le cas où $a=1$. *N. Dinculeanu* (Bucharest)

8133:

Evgrafov, M. A. The method of near systems in the space of analytic functions and its application to interpolation. Amer. Math. Soc. Transl. (2) **16** (1960), 195–314.

Translation of Trudy Moskov. Mat. Obšč. **5** (1956), 89–201 [MR **18**, 472].

8134:

Othmer, Friedrich-Ernst. Elementarer Beweis des Hauptsatzes über meromorphe Funktionenkörper. Math. Ann. **141** (1960), 99–106.

The author gives alternative simple proofs for the following fundamental theorems on the field of meromorphic functions K over a compact complex manifold $\mathfrak{P}$. Theorem 1: The meromorphic functions $f_1, \cdots, f_r$ over a compact complex manifold $\mathfrak{P}$ are analytically dependent if and only if they are algebraically dependent. Theorem 2: Let m be the maximal number of algebraically independent functions in K. For every system $f_1, \cdots, f_m$ of algebraically independent functions in K, the field K is a finite algebraic extension of the field of rational functions of $f_1, \cdots, f_m$ with respect to the complex number field C.

These results are valid also for compact complex space, but the author discusses only the case of a non-singular manifold. Evidently, they imply the following, known as Siegel's theorem: If $\mathfrak{P}$ is a compact complex n-dimensional manifold, any $n+1$ functions meromorphic over $\mathfrak{P}$ are algebraically dependent. Further, if there exist n algebraically independent functions $f_1, \cdots, f_n$ in K, K is a finite algebraic extension of the rational function field $C(f_1, \cdots, f_n)$.

The method of the proof is rather simple, and uses elementary estimations by a generalization of Schwarz's lemma, due to Siegel, in the proof of his theorem given above. *S. Hitotumatu* (Tokyo)

8135:

Kopfermann, Klaus. Periodenrelationen ausgearteter komplexer Periodentori. Math. Ann. **140** (1960), 334–343.

Let T_n be a complex torus, represented as the quotient space of the space of n complex variables modulo a lattice generated by some $n \times 2n$ complex matrix P, called the period matrix of T_n. It is known that, for the field $K(T_n)$ of meromorphic functions on the manifold T_n, the transcendence degree tdeg $K(T_n)$ of $K(T_n)$ over the field of complex numbers must satisfy tdeg $K(T_n) \leq n$ [R. Remmert, Math. Ann. **132** (1956), 277–288; MR **19**, 171]; and it is classical that tdeg $K(T_n)=n$ if and only if there is a positive definite Hermitian matrix Γ such that Im ${}^tP\Gamma\bar{P}$ is integral [C. L. Siegel, *Analytic functions of several complex variables*, Institute for Advanced Study, Princeton, N.J., 1950; MR **11**, 651]. A sketch of the proof of the latter result, closely following Siegel's version [loc. cit.], is given in this paper. Then, by using the same technique, the author shows that tdeg $K(T_n)$ is the maximum of the ranks of the positive semi-definite Hermitian matrices Γ such that Im ${}^tP\Gamma\bar{P}$ is integral. Examples are given to show that tdeg $K(T_n)$ can take all values less than n, for $n>1$ [cf. Siegel, loc. cit.]; and normal forms for the corresponding period matrices are given. *R. C. Gunning* (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 7982, 8237, 8238, 8262, 8289, 8290.

8136:

Bellman, Richard. Functional equations and theta-functions. I. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 853–854.

Introduction to a series of articles {which the author has decided not to continue}.

8137:

Moore, John T. An extension of an early result of Stieltjes. J. Soc. Indust. Appl. Math. **8** (1960), 557–559.

The concern is with 'converging factors', namely with procedures for squeezing the utmost accuracy out of asymptotic expansions, in the case of Bessel functions $J_0(a)$, $J_1(a)$ with a large positive. The remainders after n terms (where $n \sim a$) are developed in powers of $1/n$, to the terms in $1/n^2$. *T. M. Cherry* (Melbourne)

8138:

Watson, G. N. A note on Gamma functions. Proc. Edinburgh Math. Soc. (2) **11** (1958/59), Edinburgh Math. Notes No. 42 (misprinted 41) (1959), 7–9.

La formule classique de Wallis a donné lieu à des recherches de M. Kazarinoff [mêmes Notes No. 40 (1956), 19–21; MR **18**, 560] en 1956. On lui doit le résultat suivant:

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = \frac{1}{[\pi(n+\theta)]^{1/2}} \quad \text{où } \tfrac{1}{4} < \theta < \tfrac{1}{2}.$$

L'auteur retrouve ce résultat, plus d'autres, en partant de la formule relative à la fonction hypergéometrique

$$\theta(x) = -x+x\frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2(x+\frac{1}{2})} = -x+xF(-\tfrac{1}{2}, -\tfrac{1}{2}; x; 1).$$

R. Campbell (Caen)

8139:

Nanjundiah, T. S. Note on Stirling's formula. Amer. Math. Monthly **66** (1959), 701–703.

The number $\gamma_n = \log n! - \frac{1}{2}\log 2\pi - (n+\frac{1}{2})\log n + n$ possesses for large positive integral n an asymptotic expansion according to powers of n^{-1} with positive odd exponents. The remainder term is a fraction of the first neglected term [see G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Band 1, 2te Aufl., Springer, Berlin, 1954; MR **15**, 512; p. 29, problem 155], in particular $\gamma_n = 1/12n - \theta_n/360n^3$, where $0<\theta_n<1$. The author gives an elementary proof of this result by a simple refinement of the argument of Robbins [same Monthly **62** (1955), 26–29; MR **16**, 1020], who obtained the weaker result $\gamma_n = 1/12n - \theta_n/12n(12n+1)$.
J. G. van der Corput (Berkeley, Calif.)

8140:

Franz, Walter. Zur Asymptotik der Zylinderfunktionen und ihrer Nullstellen. Z. Angew. Math. Mech. **40** (1960), 385–396. (English and Russian summaries)

This paper gives asymptotic expansions for Bessel functions with complex arguments and indices, together

with the curves on which these functions become zero. Some interesting graphs are given of these curves, which are in fact the branching sections of the asymptotic expansions. This fact enables us to obtain asymptotic approximations throughout the complex planes, simply by analytic continuation. This method has the advantage that it avoids most of the difficulties which arise in the determination of those saddle points which lie on the path of integration in the Sommerfeld integral normally used.

There are some more graphs of curves of zeros with very small imaginary parts, and special sections on such indices. These arise in the discussion of Bessel functions since their zeros tend towards the origin when their index tends to a negative integer. Similar considerations apply to the Hankel functions because their zeros occur near infinity when their index is near a rational integer.

This paper gives a good outline of a very difficult part of the theory of Bessel functions.

L. J. Slater (Cambridge, England)

8141:

Popov, Blagoj S. On associated Legendre functions. Bull. Soc. Math. Phys. Macédoine **8** (1957), 5–18. (Macedonian. English summary)

The author quotes one of his earlier results [B. S. Popov, Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **10** (1957), 11–35] in which a product of two hypergeometric functions is expanded into a series of such functions, in which the coefficients contain a ${}_4F_3$ function of unit argument. On specializing the parameters, this expansion is applied to Legendre functions, and the author obtains identities due to Bailey, and Neumann and Adams. Integrals over $(-1, 1)$ of products of two and three Legendre functions are evaluated.

C. J. Bouwkamp (Eindhoven)

8142:

Leitner, Alfred; Meixner, Josef. Simultane Separierbarkeit von verallgemeinerten Schwingungsgleichungen. Arch. Math. **10** (1959), 387–391.

Beziehungen zwischen speziellen Funktionen der mathematischen Physik lassen sich gewinnen, indem man als Lösung der dreidimensionalen Schwingungsgleichung Produkte von Funktionen je einer einzigen Koordinate ansetzt. Additionstheoreme, Reihenentwicklungen und Integralbeziehungen für die betreffenden Funktionen können hieraus gefolgert werden. Für das Verfahren eignen sich nur explizit angebbare Lösungen der Schwingungsgleichung. Eine Erweiterung der Anwendungen erhält man durch ein- oder mehrmalige Differentiation dieser Lösung nach den Cartesischen Koordinaten. Auch Operationen, wie $y\partial/\partial x - z\partial/\partial y$ sind brauchbar. Eine beträchtliche Erweiterung des Bereichs der betreffenden speziellen Funktionen wird dadurch erhalten, dass die Schwingungsgleichung durch allgemeinere Gleichungen des Schrödingerschen Typs ersetzt wird. Eine solche Gleichung wird in Sphäroid-, Kugel- und Zylinderkoordinaten separiert. Hierdurch ergeben sich Anforderungen an die Gestalt der verallgemeinerten Schwingungsgleichung.

M. J. O. Strutt (Zürich)

8143:

Leitner, Alfred; Meixner, Josef. Zur Theorie der hypergeometrischen Funktionen. Arch. Math. **10** (1959), 452–459.

Zur Gewinnung von Integralbeziehungen und Additionstheoremen für die allgemeinen hypergeometrischen Funktionen wird, in Analogie zur Separation der Variablen in der Schwingungsgleichung, von folgenden verallgemeinerten Schwingungsgleichungen ausgegangen:

$$\nabla^2 u + \left(k^2 + \frac{c+dz}{x^2+y^2} + \frac{e}{z^2}\right)u = 0,$$

$$\nabla^2 u + \left(k^2 - \frac{\mu^2}{2r(r-z)} - \frac{\nu^2}{2r(r+z)}\right)u = 0.$$

Im ersten Fall wird die Gleichung in Kugelkoordinaten und Zylinderkoordinaten separiert, sowie für $k^2 = 0$ in Kugelkoordinaten und in rotationselliptischen Koordinaten; im letzten Fall in rotationselliptischen Koordinaten und in Kugelkoordinaten, falls $k^2 = 0$. Nach einer Einführung in die Bezeichnung der hypergeometrischen Funktionen werden die genannten Separationen durchgeführt. Es ergeben sich Reihenentwicklungen von separierten Wellenfunktionen, sowie Integralrelationen.

M. J. O. Strutt (Zürich)

8144:

Leitner, Alfred; Meixner, Josef. Eine Verallgemeinerung der Sphäroidfunktionen. Arch. Math. **11** (1960), 29–39.

Verfasser gehen von einer verallgemeinerten Schwingungsgleichung $\nabla^2 u + \phi(x, y, z)u = 0$ aus, welche simultan in Kugelkoordinaten und in den Koordinaten des Rotationsellipsoides mit gleichem Mittelpunkt dann und nur dann separierbar ist, wenn gilt:

$$\phi(x, y, z) = k^2 + d(x^2+y^2+z^2) + \frac{c}{x^2+y^2} + \frac{e}{z^2}.$$

Mit dieser Funktion führt die Separation in Kugelkoordinaten auf hypergeometrische Funktionen (Konfluenz mit eingeschlossen). Hierdurch können die Funktionen, welche durch Separation in rotationsellipsoidischen Koordinaten entstehen, nach hypergeometrischen Funktionen entwickelt werden. Weiter können Integralbeziehungen zwischen diesen Funktionen abgeleitet werden, deren Kern hypergeometrische Funktionen enthält. Es wird der Fall $d = 0$ und $c = 0$ betrachtet. Die Entwicklungskoeffizienten von Entwicklungsformeln der genannten Art werden abgeleitet. Dann werden spezielle Reihenentwicklungen angegeben, auch für besondere Werte der Parameter, und schliesslich werden asymptotische Formeln betrachtet.

M. J. O. Strutt (Zürich)

8145:

Blanch, Gertrude. The asymptotic expansions for the odd periodic Mathieu functions. Trans. Amer. Math. Soc. **97** (1960), 357–366.

The solutions of Mathieu's equation

$$y'' + (a - 2q\cos 2x)y = 0$$

which possess period 2π are examined for large positive values of the parameter q. It is found that asymptotically

$$ce_n(x, q) = C_n[Z_{0,n} + Z_{1,n}],$$

$$Se_{n+1}(x, q) = S_n[Z_{0,n} - Z_{1,n}]\sin x,$$

where $Z_{0,n}$ and $Z_{1,n}$ can be expressed as series in the functions of the parabolic cylinder $D_{n+4k}(\alpha)$ and $D_{n+4k+2}(\alpha)$, respectively ($k = 0, \pm 1, \pm 2, \cdots$; $\alpha = 2q^{1/4}\cos x$). Previous

results in the literature are thus corrected and a considerable improvement in the numerical results is obtained. A set of useful formulas is given in the Appendix, in particular, expressions for $Z_{0,n}$ and $Z_{1,n}$ $(n=0, 1)$, expansions in powers of $q^{-1/2}$ of C_n, S_n and of expressions like $ce_0(0, q)/ce_0(\pi/2, q)$. *J. Meixner* (Aachen)

8146:

Beard, R. E. An integral related to the confluent hypergeometric function. J. Inst. Actuar. **86** (1960), 296–307.

In this paper, the problem of the numerical evaluation of the function

$$I(p, q; \alpha) \equiv B(p+1, q+1)M(p+1; p+q+2; \alpha)$$

is considered, where M is a confluent hypergeometric function, and B is a beta function. The main formulae are summarised, including recurrence relations, and integrals, an expansion as a series of incomplete gamma functions, and some asymptotic expansions.

In particular, the paper includes some useful expansions of $I(p, q; \alpha)$ for p and $q>1$, near the turning points of the integrand $e^{\alpha t}t^p(1-t)^q$, and concludes with several illuminating numerical examples, and a note on the incomplete form of the function. *L. J. Slater* (Cambridge, England)

8147:

Singh, V. N. Certain expansions involving E-functions. Proc. Glasgow Math. Assoc. **3**, 119–122 (1957).

$$\frac{\Gamma(\alpha_2+n)}{\Gamma(n)} E(p; \alpha_r: q; \rho_s: z)$$

is expressed as an infinite sum of E-functions multiplied by ratios of gamma functions and negative integral powers of z. Series expansions for products of certain ${}_pF_q$'s and E-functions in terms of multiple series of E-functions are also derived. *N. D. Kazarinoff* (Moscow)

8148:

MacRobert, T. M. Integration of E-functions with respect to their parameters. Proc. Glasgow Math. Assoc. **4**, 84–87 (1959).

For example, if $p \geqq q$, $|\arg z| < \frac{1}{2}(p-q+2)\pi$, $\Re(\alpha_n)>0$ $(n=1, \cdots, p)$,

$$\frac{1}{2\pi i}\int \Gamma(\zeta)\Gamma(\beta-\zeta)E(\gamma, \alpha_1-\zeta, \cdots, \alpha_p-\zeta: q; \rho_s-\zeta: z)z^\zeta\, d\zeta = B(\beta, \gamma)E(\beta+\gamma, \alpha_1, \cdots, \alpha_p: q; \rho_s: z).$$

The results can be used to sum series of products of E-functions. *N. D. Kazarinoff* (Moscow)

8149:

MacRobert, T. M. Infinite series of E-functions. Proc. Glasgow Math. Assoc. **4**, 26–28 (1958).

Several series of E-functions are summed by expressing the E-functions as Barnes integrals, interchanging the order of summation and integration, and using evaluations of ${}_{p+1}F_p$'s at ± 1. The sums are also E-functions. *N. D. Kazarinoff* (Moscow)

8150:

MacRobert, T. M. Applications of the multiplication formula for the gamma function to E-function series. Proc. Glasgow Math. Assoc. **4**, 114–118 (1960).

The author has previously used the Barnes integral to sum series of E-functions; see #8149. He now adds the multiplication formula for the gamma function to his tool kit and obtains sums for series involving other types of E-functions. *N. D. Kazarinoff* (Moscow)

8151:

Popov, B. S. Sur les polynomes de Legendre. Mathesis **68** (1959), 239–242.

The identity

$$\int_{-1}^{1} P_m(x)P_n(x)\cdots P_r(x)P_{m+n+\cdots+r}(x)\, dx = \frac{A_mA_n\cdots A_r}{A_{m+n+\cdots+r}}\,\frac{2}{2(m+n+\cdots+r)+1},$$

where $A_m = 1\cdot 3\cdots(2m-1)/m!$, $m>0$, $A_0=1$, and $P_m(x)$ is the Legendre polynomial of order m, is established for any non-negative integers $m, n, \cdots, r$. *A. E. Danese* (Schenectady, N.Y.)

8152:

Chatterjea, S. K. Certain identities and inequalities concerning Hermite polynomials. Boll. Un. Mat. Ital. (3) **15** (1960), 25–29.

En introduisant la fonction

$$\Delta_{n,h,k;x}(H) = H_{n+h}(x)H_{n+k}(x) - H_n(x)H_{n+k+h}(x),$$

où n, h, k sont entiers, égaux ou supérieurs à 1, x réel, l'auteur établit des nouvelles identités relatives aux polynomes d'Hermite et retrouve l'inégalité de Turán. *R. Campbell* (Caen)

8153:

Eweida, M. T. On Bessel polynomials. Math. Z. **74** (1960), 319–324.

The author uses the notation

$$Y_n^{(\alpha)}(x) = y_n(x, \alpha+2, 2), \quad Y_n^{(0)}(x) = y_n(x),$$

$$\Theta_n^{(\alpha)}(x) = x^n Y_n(1/x), \quad \theta_n(x) = x^n y_n(1/x),$$

where $y_n(x, a, b)$ is the notation of Krall and Frink [Trans. Amer. Math. Soc. **65** (1949), 100–115; MR **10**, 453]. Among the results obtained in the present paper the following may be cited:

$$\theta_n(x) = \frac{x^{2n+1}}{2^n n!}\, e^x \int_1^\infty e^{-xt}(t^2-1)^n\, dt,$$

$$\theta_n(x) = 2^{n+1}\pi^{-1}n!e^x x^{2n+1}\int_0^\infty \frac{\cos t}{(t^2+x^2)^n}\, dt,$$

$$\pi\theta_n^2(x) = 4x^{2n+1}e^{2x}\int_0^\infty K_0(2x\cosh t)\cosh(2n+1)t\, dt,$$

$$\theta_{n-1}(x) = x^{2n-1}e^x\int_x^\infty e^{-t}t^{-2n}\theta_n(t)\, dt,$$

$$\theta_n(u)\theta_n(v) = (2\pi)^{-1/2}e^{u+v} \times \int_0^\infty \exp\left\{-\tfrac{1}{2}\left(t+\frac{(u+v)^2}{t}\right)\right\} t^{n-1/2}\theta_n\left(\frac{uv}{t}\right) dt,$$

$$\Theta_n^{(\alpha)}(\lambda x) = \lambda^{2n+\alpha+1} \sum_{k=0}^{\infty} \frac{(n+\alpha+1)_k}{k!} (1-\lambda)^k \Theta_n^{(\alpha+k)}(x).$$

The last may be compared with Al-Salam's multiplication formula for $Y_n^{(\alpha)}(x)$ [Duke Math. J. **24** (1957), 529–545; MR **19**, 849; formula (8.2)]. *L. Carlitz* (Durham, N.C.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 8111, 8519.

8154:

Laitoch, Miroslav. Contribution aux transformations des solutions d'équations différentielles linéaires. Czechoslovak Math. J. **10 (85)** (1960), 258–270. (Russian. French summary)

Consider the differential equations (ω): $y''+2\omega y'+\omega' y=0$, (Ω): $Y''+2\Omega Y'+\Omega' Y=0$ where ω' and Ω' are continuous on suitable intervals. With these equations one associates (Ω,ω): $F(X)+\frac{1}{2}\Omega(X)X'^2=\frac{1}{2}\omega$, where $F(z)=\frac{1}{2}z'''/z'-\frac{3}{4}z''^2/z'^2$. Then we have the following theorems. (1) If U is a solution of (Ω) and X a solution of (Ω,ω) satisfying $X(t_0)=X_0$, $X'(t_0)=X_0'\neq 0$, $X''(t_0)=X_0''$, then $u=U(X)/X'$ satisfies (ω) for certain initial conditions. (2) The solution U, of (Ω), in (1) satisfies the functional equation $U=u(x)/x'$, where x and X are inverses of each other. *S. Hoffman* (Hartford, Conn.)

8155:

Moser, Jürgen. The order of a singularity in Fuchs' theory. Math. Z. **72** (1959/60), 379–398.

Consider a system of differential equations

$$(*) \qquad dy/dx = A(x)y,$$

where y is an n-vector, $A(x)$ an $n\times n$ matrix, and the elements of $A(x)$ are analytic functions of the complex variable x in a disk $|x|<\rho$ except for poles of order $\leqq p$ $(p\geqq 1)$ at $x=0$. $A(x)$ thus has an expansion

$$A(x) = x^{-p}\sum_{\nu=0}^{\infty} A_\nu x^\nu, \quad A_0\neq 0,$$

where the A_ν are constant matrices. If the eigenvalues of A_0 are all distinct, the behavior of the solutions of (*) near $x=0$ can be described in terms of p; for instance, $p=1$ is a necessary and sufficient condition for $x=0$ to be a regular singular point in the sense of Fuchs. If no such assumption is made about A_0, a transformation (**) $y=T(x)z$ (where $T(x)$ is a matrix whose elements are analytic near $x=0$ except for possible poles at $x=0$) may carry (*) into a system $dz/dx=B(x)z$ with a different value of p.

The author addresses himself to the problem of developing a simple criterion which, for a given system, makes it possible to decide whether there exists a transformation (**) which reduces the value of p, or, rather, of a quantity $m(A)$ which reduces to p if the rank r of A_0 is equal to n. This quantity—the order of A—is defined as $m(A)=p-1+r/n$. The author obtains a necessary and sufficient criterion for the reducibility of $m(A)$ which depends only on A_0 and A_1. He shows furthermore that such a reduction, if it is possible, can be carried out by a finite number of transformations (**) with matrices $T(x)$ of a particularly simple form. *Z. Nehari* (Pittsburgh, Pa.)

8156:

Temple, G. Linearization and delinearization. Proc. Internat. Congress Math. 1958, pp. 233–247. Cambridge Univ. Press, New York, 1960.

This paper surveys briefly the problems of solving nonlinear differential equations. Two methods are discussed. One is the method of linearization by which non-linear equations are forcibly reduced to an associated, approximate linear form. The other is the method of delinearization by which the non-linearities are partially restored. For each case, examples are cited. *Y. H. Kuo* (Peking)

8157:

Marhašov, L. M. On the characteristic exponents of the solutions of a second order linear differential equation with periodic coefficients. Prikl. Mat. Meh. **23** (1959), 1066–1073 (Russian); translated as J. Appl. Math. Mech. **23**, 1525–1535.

In the equation $\ddot{x}+q_1(t)\dot{x}+q_2(t)x=0$, q_1, q_2 are continuous and periodic with period ω. If the characteristic exponents of the equation are λ_1, λ_2 then it is known that

$$\omega^{-1}\int_0^\omega q_1(t)\,dt = -(\lambda_1+\lambda_2),$$

that is, the particular function $q_1(t)$ has mean value $-(\lambda_1+\lambda_2)$ over a period.

The author remarks that if there were another function of q_1 and q_2 with mean value over a period expressible as a function of λ_1 and λ_2, but essentially different from $(\lambda_1+\lambda_2)$, then λ_1 and λ_2 could be calculated by a finite number of operations on q_1 and q_2. The remainder of the paper shows that this is in fact impossible; the mean value of any such function is of the form $\mu_1(\lambda_1+\lambda_2)+\mu$, where μ, μ_1 are constants. *F. M. Arscott* (Battersea)

8158:

Banks, Dallas. Bounds for the eigenvalues of some vibrating systems. Pacific J. Math. **10** (1960), 439–474.

The Sturm-Liouville problem

$$(ru')'+[\lambda p-q]u=0,\quad u'(0)-h_0u(0)=0,$$
$$u'(1)+h_0u(1)=0,$$

$$r>0,\quad q\geqq 0,\quad p\geqq 0,\quad \int_0^1 p\,dx=1,\quad h_0\geqq 0,\quad h_1\geqq 0$$

is considered. The author shows that among monotone density functions p the minimum value of λ_1 and, more generally, the maximum value of $\sum_1^n \lambda_i^{-1}$ are attained when $p=0$ for $x<t_0$ and $p=(1-t_0)^{-1}$ for $x>t_0$. The number t_0 is the solution of a transcendental equation depending upon n. The minimum value of λ_n is attained for a step function p with n jumps.

Among convex density functions p the minimum of $\sum_1^n \lambda_i^{-1}$ is attained when $p=0$ for $x<t_0$, $p=2(x-t_0)\times(1-t_0)^{-2}$ for $x>t_0$ for a suitable t_0. The minimum of λ_n occurs for a $p(x)$ which is continuous and piecewise linear with at most n jumps in its derivative. Similar results hold for concave p.

Similar results are also obtained for the vibrating rod and the vibrating membrane.

H. F. Weinberger (Minneapolis, Minn.)

8159:

Howard, Henry. Oscillation criteria for fourth-order linear differential equations. Trans. Amer. Math. Soc. **96** (1960), 296–311.

The author uses the technique of Leighton and Nehari [same Trans. **89** (1958), 325–377; MR **21** #1429] to derive oscillation criteria for the equation (1) $(r(x)y'')''=p(x)y$. The main result concerns an eigenvalue problem slightly different from the one considered by Leighton and Nehari, namely

$$(2) \qquad (r(x)u'')'' = \lambda p(x)u,$$
$$u(a) = u'(a) = r(b)u''(b) = (r(x)u''(x))'_b = 0, \quad 0 < a < b.$$

It is proved that, if $\int_a^\infty dx/r(x) < \infty$, (1) is nonoscillatory in (a, ∞) if, and only if, $\lambda_b > 1$ for any $b > a$, where λ_b is the first eigenvalue of (2). By means of this result the author derives both necessary conditions for the non-oscillation of the solutions of (1) and the oscillation and nonoscillation criteria for (1). The same technique may also be applied to second-order equations. Only examples of these theorems, without proofs, are given.

M. Zlámal (Brno)

8160:

Kemp, R. R. D. On a class of non-self-adjoint differential operators. Canad. J. Math. **12** (1960), 641–659.

In a previous paper [same J. **10** (1958), 447–462; MR **20** #1817] the author considered the eigenfunction expansion theorem associated with a non-self-adjoint differential operator of the second order. This paper contains a generalization to nth order differential operators. A Green's function is constructed and from this an expansion formula is obtained for a suitably restricted class of functions.

E. C. Titchmarsh (Oxford)

8161:

Dorodnicyn, A. A. Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order. Amer. Math. Soc. Transl. (2) **16** (1960), 1–101.

Translation of Uspehi Mat. Nauk **7** (1952), no. 6 (52), 3–96 [MR **14**, 876].

8162:

Naĭmark, M. A. Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis. Amer. Math. Soc. Transl. (2) **16** (1960), 103–193.

Translation of Trudy Moskov. Mat. Obšč. **3** (1954), 181–270 [MR **15**, 959].

8163:

Krasnosel'skiĭ, M. A. On the possibility of generalizing the method of orthogonal trajectories. Amer. Math. Soc. Transl. (2) **16** (1960), 373–374.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 160–162 [MR **20** #2638].

8164:

Fage, M. K. Construction of transformation operators and the solution of a problem of moments for ordinary linear differential equations of arbitrary order. Amer. Math. Soc. Transl. (2) **16** (1960), 457–462.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 240–245 [MR **19**, 273].

8165:

Lyubič, Yu. I. Conditions for the uniqueness of the solution to Cauchy's abstract problem. Dokl. Akad. Nauk SSSR **130** (1960), 969–972 (Russian); translated as Soviet Math. Dokl. **1**, 110–113.

The vector function $x(t)$ $(t \geq 0)$ is called a weak solution of the abstract Cauchy problem:

$$\frac{dx(t)}{dt} = Ax(t) \quad (t > 0), \qquad x(0) = x_0,$$

if it is weakly absolutely continuous and almost everywhere weakly differentiable for $t > 0$, weakly continuous at $t = 0$ and satisfies the equation almost everywhere. It is shown that there is at most one weak solution if the resolvent $R(\lambda, A)$ exists on a half-line and

$$\sigma = \limsup_{\lambda\to\infty} \lambda^{-1} \log \|R(\lambda, A)\|$$

is finite. A counterexample is constructed to show that if σ is not finite uniqueness may not hold, using an $L_\alpha^2(0, \infty)$ space with a suitably constructed weight function α, the operator A corresponding to differentiation.

A. V. Balakrishnan (Los Angeles, Calif.)

8166a:

Opial, Zdzislaw. La presque-périodicité et les trajectoires sur le tore. C. R. Acad. Sci. Paris **250** (1960), 3565–3566.

8166b:

Denjoy, Arnaud. Sur les trajectoires du tore. C. R. Acad. Sci. Paris **251** (1960), 175–177.

The first paper presents a proof of the uniform almost periodicity of the minimal flow on a torus given by $d\theta/d\varphi = A(\varphi, \theta)$, where $A(\varphi, \theta)$ is continuous and of period one in each variable, and every trajectory is dense. The method is to use the map onto the standard irrotational flow. The second paper gives an alternative proof, and estimates the translation numbers. {Notice that these results do not contradict Markov's example 8.19 in Nemyckiĭ and Stepanov, *Kačestvennaya teoriya differencial'nyh uravneniĭ* [OGIZ, Moscow-Leningrad, 1947; MR **10**, 612]; in the present case the map onto the group translation system is uniform in time, too.}

L. W. Green (Minneapolis, Minn.)

8167:

Fleishman, B. A. Harmonic and subharmonic response of an on-off control system to sinusoidal inputs. J. Franklin Inst. **270** (1960), 99–113.

The second-order, piecewise-linear, ordinary differential equation (1) $\ddot{x}+x+\operatorname{sgn} x=-\ddot{y}$, with $y=y_0 \sin \omega t$, is considered. (1) is written as (2) $\ddot{x}_f+x_f=-\ddot{y}$ and (3) $\ddot{x}_1+x_1=-\operatorname{sgn} x_1$. Then, if $\operatorname{sgn} x_h=\operatorname{sgn} x_1$, $x_h=x_f+x_1$ is the solution of (1). By means of a phase-plane analysis it is shown that, for $\omega>1$, x_h is a harmonic solution of (1) if $y_0<0$; and that, if $y_0>0$, x_h is still a harmonic solution if $x_f+x_1>0$ for $0<t<\pi/\omega$. The necessary and sufficient conditions for

a subharmonic solution of integral order m, $\omega > m > 1$, are also given. In an appendix it is shown how a simple control system involving a relay may have (1) as its equation of behavior. *N. H. Choksy* (Silver Spring, Md.)

8168:

Sansone, Giovanni. Soluzioni periodiche di seconda specie dell'equazione del pendolo generalizzata. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1959), 59–79.

L'A. studia l'equazione del pendolo generalizzata:

$$(1) \qquad \frac{d^2\vartheta}{dt^2}+f(\vartheta, \alpha)\frac{d\vartheta}{dt} = g(\vartheta),$$

e la corrispondente equazione delle caratteristiche

$$(2) \qquad \frac{dz}{d\vartheta} = \frac{g(\vartheta)}{z}-f(\vartheta, \alpha) \qquad (z = \dot{\vartheta}),$$

ove α è un parametro >0. Si hanno, tra l'altro, i seguenti enunciati.

Nelle ipotesi: (i) $f(\vartheta, \alpha)>0$, definita per $\alpha>0$, $-\infty<\vartheta<+\infty$, continua e periodica, con periodo 2π, come funzione di ϑ, crescente come funzione di α; (ii) $g(\vartheta)$ lipschitziana e periodica, di periodo 2π; (iii) $\int_0^{2\pi} g(\vartheta)\,d\vartheta>0$, esiste un valore critico α_0 (che può essere anche nullo o infinito) tale che l'equazione (2) ammetta una soluzione periodica di seconda specie per $\alpha<\alpha_0$, non ne ammetta per $\alpha>\alpha_0$.

Se si ammette, in più, che: la derivata $\partial f/\partial\vartheta$ sia continua rispetto a ϑ; sia $\lim_{\alpha\to\infty} f(\vartheta, \alpha)=\infty$ uniformemente rispetto a ϑ; la derivata $g'(\vartheta)$ sia continua e $g(\vartheta)$ possegga, in $[0, 2\pi]$, solo zeri semplici, allora la (2) ammette una e una sola soluzione periodica di seconda specie.

L. Amerio (Milan)

8169:

Giertz, M. The solution of a problem of stability in periodic systems with methods from the calculus of variations. Kungl. Tekn. Högsk. Handl. Stockholm No. 150 (1959), 10 pp.

This paper solves the following variation problem: Let $Y(t)$ be a vector with p square integrable components $(0\leqq t\leqq 2\pi)$ and Λ_0 a given constant p by p matrix. The problem is to minimize the integral $\int_0^{2\pi} |Y'+\Lambda_0 Y|^2 dt$ under the boundary condition $Y(2\pi)=e^{2\pi i x}Y(0)$ and the normalization $\int_0^{2\pi} |Y|^2 dt=1$. The solution is given in terms of the Fourier expansions and discussed for a special 2 by 2 matrix Λ_0. The problem arises from the study of the stability region of systems of differential equations with periodic coefficients as was suggested by Göran Borg [Actes du Colloque Internat. des Vibrations non linéaires (Ile de Porquerolles, 1951), pp. 21–29, Publ. Sci. Tech. Ministère de l'Air, No. 281, Paris, 1953; MR **15**, 223].

J. Moser (New Rochelle, N.Y.)

8170:

Lillo, James C. A note on the continuity of characteristic exponents. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 247–250.

The author points out that the characteristic numbers of Lyapunov and Perron for the system (1) $dx/dt=A(t)x$, A continuous and bounded, are not upper semi-continuous with respect to the uniform norm of the coefficient matrix (i.e., $\|A\|=\sup\|A(t)\|$, $t\in R$, where R is the real line, or the positive half line). This motivates his introduction of a new number, the major characteristic exponent $\lambda(A)$ defined by

$$\lambda(A) = \limsup_{t\to\infty}\ [\sup_{\varphi(t),t_0\in R}\{\ln(\|\varphi(t_0+t)\|/\|\varphi(t_0)\|)/t\}],$$

where φ is a non-trivial solution of (1). The main theorem then states that for systems of the form (1), with A bounded, the major characteristic exponent $\lambda(A)$ is upper semi-continuous. As an application the author remarks that if $\lambda(A)<0$, then for any $\varepsilon>0$ there exists a continuous matrix $P(\varepsilon, t)$, with P, P^{-1}, and dP/dt bounded, such that the transformation $x=Py$ transforms (1) into $dy/dt=C^*(t)y+B(t)y$, where $\|B\|<\varepsilon$, and $C^*(t)$ is a diagonal matrix such that a fundamental matrix solution $Y(t)$ of $dy/dt+C^*(t)y$ exists satisfying $\|Y(t)Y^{-1}(s)\|\leqq \beta e^{-\alpha(t-s)}$, for β, $\alpha>0$, $t\geqq s\in R$. This remark is then applied to (2) $dx/dt=A(t)x+f(x, \varepsilon, t)$, where A is almost periodic in t, and f is almost periodic in t uniformly with respect to x, for x in some open subset of R^n containing the origin. The theorem obtained then states that if $\lambda(A)<0$, and $\|f(x, \varepsilon, t)\|=O(\|x\|^2+|\varepsilon|)$ as $\|x\|$, $|\varepsilon|\to 0$, then for ε sufficiently small there exists a stable almost periodic solution of (2), $p(\varepsilon, t)$, unique in a certain neighborhood of 0, such that $p(\varepsilon, t)\to 0$ as $\varepsilon\to 0$ uniformly on R.

For further results, see Lillo, Acta Math. **103** (1960), 123–138 [MR **22** #1722] and #8171.

A. Stokes (Baltimore, Md.)

8171:

Lillo, James C. Continuous matrices and the stability theory of differential systems. Math. Z. **73** (1960), 45–58.

Consider the linear differential system (2.1) $\dot{x}=B(t)x$ where $B(t)$ is a matrix in M_n, that is, $B(t)$ is a complex $n\times n$ matrix which is continuous on $0\leqq t<\infty$ and uniformly bounded, $\|B\|=\sup_{0\leqq t<\infty}\|B(t)\|<\infty$ where $\|B(t)\|=\sum_{i,j}|b_{ij}(t)|$ for each $t\geqq 0$.

A vector solution $x(t)$ is said to have a strong characteristic exponent λ in case

$$\lim_{t\to\infty} \log\frac{\|x(t)\|}{t} = \lambda.$$

Theorem: A necessary and sufficient condition that the system (2.1) possess n distinct strong characteristic exponents $[\lambda_1, \cdots, \lambda_n]$ is that it be kinematically similar to a super triangular system whose diagonal elements possess the mean values $[\lambda_1, \cdots, \lambda_n]$.

The author also defines $A(t)\in M_n$ to be approximately similar to $B(t)\in M_n$ in case for each $\varepsilon>0$ there exists a $P(\varepsilon, t)$ with P, P^{-1}, $\dot{P}\in M_n$, and

$$\|P^{-1}(-\dot{P}+A(t)P)-B(t)\| < \varepsilon.$$

The author then applies his results on linear differential systems to the problem of the existence of a stability manifold near a critical point of a nonlinear differential system and to the problem of perturbing a bounded solution of a nonlinear differential system, upon a change in a coefficient parameter.

L. Markus (Minneapolis, Minn.)

8172:

Mufti, I. H. Stability in the large of autonomous systems of two differential equations. Arch. Rational Mech. Anal. **6**, 115–132 (1960).

The author discusses the global asymptotic stability of the origin for the real differential system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y),$$

where $f(x, y)$, $g(x, y)$ are continuous in the plane, uniqueness of solutions of the initial value problem is assumed, and $f(0, 0) = g(0, 0) = 0$. Typical results are as follows.

For the system

$$\dot{x} = ax + f(y), \qquad \dot{y} = bx + cy,$$

with $f(0) = 0$, $a + c < 0$, $ac - bh(y) > 0$ for $y \neq 0$, where $h(y) = f(y)/y$ for $y \neq 0$, the origin is globally asymptotically stable.

For the system

$$\dot{x} = ax + f_1(y), \qquad \dot{y} = f_2(x) + cy,$$

with $a + c < 0$, $ac - h_1(y)h_2(x) > 0$ for $x \neq 0$, $y \neq 0$, $f_1(0) = f_2(0) = 0$, where $f_1(y) = yh_1(y)$, $f_2(x) = xh_2(x)$, assume $h_1(y) > 0$ for $y \neq 0$ and $h_2(x) > 0$ for $x \neq 0$ (or $h_1(y) < 0$ for $y \neq 0$ and $h_2(x) < 0$ for $x \neq 0$). Then the origin is globally asymptotically stable.

The methods use the geometry of the phase plane and the construction of Lyapunov functions.

L. Markus (Minneapolis, Minn.)

8173:

Ezeilo, J. O. C. On the stability of solutions of certain differential equations of the third order. Quart. J. Math. Oxford Ser. (2) **11** (1960), 64–69.

The author considers the third-order differential equation

$$(1) \qquad \dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0,$$

where $f(x, y)$, $\partial f(x, y)/\partial x$, $g(y)$ and $h'(x)$ are continuous functions; using an appropriate Lyapunov function, he proves the following theorem. Suppose that $g(0) = h(0) = 0$ and that (i) $f(x, y) \geq \delta_0 > 0$, for all x, y; (ii) $g(y)/y \geq \delta_1 > 0$ $(y \neq 0)$, $h(x)/x \geq \delta_2 > 0$ $(x \neq 0)$; (iii) $h'(x) \leq c$, for all x, where $\delta_0\delta_1 - c > 0$; (iv) $y\partial f(x, y)/\partial x \leq 0$, for all x, y. Then every solution $x(t)$ of (1) satisfies $x(t) \to 0$, $\dot{x}(t) \to 0$, $\ddot{x}(t) \to 0$, as $t \to \infty$. He proves also that that part of (ii) which concerns $h(x)$ can be replaced by $h(x)/x > 0$ $(x \neq 0)$, $H(x) = \int_0^x h(\xi)d\xi \to \infty$, as $|x| \to \infty$.

Z. Opial (Kraków)

8174:

Borg, Göran. A condition for the existence of orbitally stable solutions of dynamical systems. Kungl. Tekn. Högsk. Handl. Stockholm No. 153 (1960), 12 pp.

This paper gives a criterion for existence, uniqueness and stability of a periodic solution of a system of differential equations $\dot{y} = f(y)$ (where y and f are n-vectors). To state the condition explicitly we denote by $V(y)$ the Jacobian matrix of $f(y)$. In a bounded domain D it is assumed that $\|V(y)\| \leq K$, $\delta \leq |f(y)| \leq K$, $V(y)$ uniformly continuous, and $(y - z, V(y)(y - z)) \leq -\delta|y - z|^2$ provided $(f(y), y - z) = 0$, with some $\delta > 0$, $K > 0$. If, in addition, there exists a solution $y(t)$ which remains in D for $t > 0$, then there exists a unique periodic solution in D. The proof is geometric, and ensures, moreover, that for any two solutions in D which are sufficiently close for $t = 0$, the "normal" distance approaches zero as $t \to \infty$.

J. Moser (New Rochelle, N.Y.)

8175:

Bogusz, Wladyslaw. Application of the retract method in non-linear engineering problems. Arch. Mech. Stos. **12** (1960), 437–450. (Polish and Russian summaries)

The present paper is a very good explanation, suitable for engineers, of Wazewski's method for discussing the asymptotic behavior of the solutions of ordinary differential equations. The paper contains many examples. The author uses the terms issue and rigorous issue for the more standard English terms egress and strict egress, respectively. It is not clear what the author means by a surface, but it is probably a function $z = f(x)$.

J. K. Hale (Baltimore, Md.)

8176:

Heading, J. The Stokes phenomenon and certain nth-order differential equations. III. Matrix applications. Proc. Cambridge Philos. Soc. **56** (1960), 329–341.

In two earlier papers [same Proc. **53** (1957), 399–418, 419–441; MR **19**, 140] the author investigated asymptotic solutions of the differential equation $d^n u/dz^n = (-1)^n z^m u$, $m \neq n$. He now rewrites this equation as a system of n first order equations, in matrix form $e' = Te$, where e is a column vector with components $e_i = u^{(i-1)}$, $i = 1, 2, \cdots, n$, $e' = de/dz$, and T is a square matrix. He then determines R so that $R^{-1}TR$ is diagonal: $R = ZK$, where Z is the diagonal matrix with diagonal elements

$$z_{ii} = (-1)^{i-1} z^{(i-1)m/n} \qquad (i = 1, 2, \cdots, n)$$

and K is a constant matrix, and considers the vector differential equation for $f = R^{-1}e$. He obtains and discusses asymptotic solutions of the differential equation for f, discusses the Stokes phenomenon, and determines the Stokes matrices which effect the change in the constants consequent upon crossing a Stokes line.

A. Erdélyi (Pasadena, Calif.)

8177:

Kruskal, Martin. Asymptotic theory of systems of ordinary differential equations with all solutions nearly periodic. La théorie des gaz neutres et ionisés (Grenoble, 1959), pp. 275–284. Hermann, Paris; Wiley, New York; 1960.

Consider the system (1) $x' = f(x, \varepsilon)$, where x is an n-vector and ε is a real parameter. Suppose that all of the solutions of $x' = f(x, 0)$ are nonconstant and periodic in some region Ω of x-space. The author claims to have a procedure (generalizing slightly the method of Krylov and Bogoliubov) for obtaining asymptotic series for the solutions of (1) which are valid for a range of t of order $1/\varepsilon$ and $x \in \Omega$. The proof of this assertion is not given in the paper.

J. K. Hale (Baltimore, Md.)

8178:

Mlak, W. Integration of differential equations with unbounded operators in abstract (L)-spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 163–168. (Russian summary, unbound insert)

Let E be an (L)-space and let S be the positive cone of S. Let $A(t)$ be infinitesimal generators of semi-groups of contractions so that $D = S \cap D(A(t))$ is dense in S. Suppose that $0 \leq (I - \varepsilon A(t))^{-1}x \leq x$ for all $x \in S$, that $(I - \varepsilon A(t))^{-1}x$ is continuous in $t \in (0, a)$ and that $|A(t)x|$ is (for all $x \in D$) integrable over $(0, a)$. The author constructs

a non-stationary "process" (i.e., $U(t, s)U(s, u) = U(t, u)$ for $0 \leqq u \leqq s \leqq t$, $U(t, t) = I$, $\partial U(t, s)x/\partial t = A(t)U(t, s)x$ for $x \in D$ almost everywhere in $t \in (s, a)$, etc.) satisfying the differential equation $du/dt = A(t)u$. *C. Foiaş* (Bucharest)

8179:

Borisovič, Yu. G. A weak topology and periodical solutions of differential equations. Dokl. Akad. Nauk SSSR **136** (1961), 1269–1272 (Russian); translated as Soviet Math. Dokl. **2**, 176–179.

Using the notions of weak topology and rotation of a vector field the author obtains sufficient conditions for the existence of periodic solutions of the B-space differential equation $dx/dt = F(t, x)$.

J. G. Wendel (Ann Arbor, Mich.)

8180:

Kukles, I. S.; Suyaršaev, A. M. Frommer's generalized method. Dokl. Akad. Nauk SSSR **136** (1961), 29–32 (Russian); translated as Soviet Math. Dokl. **2**, 20–23.

Consider the differential equation

$$(1) \qquad dy/dx = \sum_{i=0}^{m} \alpha_i y^{m-i} \Big/ \sum_{j=0}^{n} \beta_j y^{n-j},$$

where α_0 and β_0 are constants ($\alpha_0^2 + \beta_0^2 \neq 0$), the functions $\alpha_i(x)$ ($i = 1, \cdots, m$) and $\beta_j(x)$ ($j = 1, \cdots, n$) are differentiable for small $x > 0$, conserve their signs and vanish with x. The problem is to examine whether there exist characteristics of equation (1), entering the origin from the right; and, if they exist, whether the set of such characteristics is finite or infinite. Also the problem of estimating the order of smallness of the solutions $y(x)$ of (1), vanishing at the origin, is discussed. If $\alpha_i(x)$ and $\beta_j(x)$ are analytic, the above problems are solved by Frommer's method [Math. Ann. **99** (1928), 222–272; cf. also I. S. Kukles, same Dokl. **117** (1957), 367–370; MR **21** #7337]. In the present more general case use is made of a more general method, of which Frommer's method is a particular case. For the details of the method and results the reader is referred to the paper itself.

E. Leimanis (Vancouver, B.C.)

8181:

Massera, J. L.; Schäffer, J. J. Linear differential equations and functional analysis. IV. Math. Ann. **139**, 287–342 (1960).

[For part III see Ann. of Math. (2) **69** (1959), 535–574; MR **21** #3638.] Let X be a Banach space, $J = [0, \infty)$, and suppose $t \to A(t)$ is a mapping of J into the Banach space of continuous linear operators of X into itself which is uniformly Bochner integrable on every bounded subinterval of J. A closed linear subspace Y of X is said to induce a dichotomy [resp. exponential dichotomy] of the solutions of (I) $x' + A(t)x = 0$ if there exist positive constants $\gamma < 1$, γ_0, N_i [and ν_i], $i = 0, 1$, such that: $x(0) \in Y$ implies, for $t \geqq t_0 \geqq 0$,

$$(1) \quad \|x(t)\| \leqq N_0 \|x(t_0)\| \quad [\text{resp. } \|x(t)\| \leqq N_0 \exp(-\nu_0(t - t_0))\|x(t_0)\|];$$

$x(0) \in' Y$ and $\gamma[Y, x(0)] \geqq \gamma$ imply, for $t \geqq t_0 \geqq 0$,

$$(2) \quad \|x(t)\| \geqq N_1 \|x(t_0)\| \quad [\text{resp. } \|x(t)\| \geqq N_1 \exp(\nu_1(t - t_0))\|x(t_0)\|];$$

$x_0(0) \in Y$, $x_1(0) \in' Y$, $x_i(0) \neq 0$, and $\gamma[Y, x_1(0)] \geqq \gamma$ imply $\gamma[x_0(t), x_1(t)] \geqq \gamma_0$ for $t \geqq 0$. Here $\gamma[y, x] = \|y/\|y\| - x/\|x\|\|$ and $\gamma[Y, x] = \inf\{\gamma[y, x] : y \in Y - \{0\}\}$ for $x \in' Y$. If $Y = \{0\}$ it is only required that $x(0) \in' Y$ imply (2).

In part I [ibid. **67** (1958), 517–573; MR **20** #3466] the authors proved, among others, the following result concerning the linear manifold $X_0 \subset X$ of initial points $x(0)$ of bounded solutions of (I). If X_0 is closed and has a closed complement, then X_0 induces a dichotomy if and only if (II) $x' + A(t)x = f(t)$ has at least one bounded solution for every $f \in L^1$; and X_0 induces an exponential dichotomy if and only if (II) has at least one bounded solution for every $f \in L^p$, $1 < p \leqq \infty$, provided (*) $\sup_{t \in J} \int_t^{t+1} \|A(s)\| ds < \infty$.

In the present part IV the authors attack the question as to when a closed linear subspace of X induces an ordinary or exponential dichotomy in a much more general setting and from a more unified point of view. Their new results represent not only considerable extensions and refinements of their earlier ones; also they concern new types of stability properties, less local in nature than (1), (2), in that they are formulated in terms of different norms and mean values over certain "slices" of the solutions of (I). Only a bare outline of the principal results can be given here.

Let B, D be Banach spaces of mappings of J into X such that convergence in either implies convergence in the mean on every bounded subinterval of J. The authors call the pair (B, D) admissible if, for every $f \in B$, (II) has at least one solution which belongs to D. The pair (B_1, D_1) is stronger than (B_2, D_2) if B_2, D_1 are stronger than B_1, D_2, respectively. A pair (B, D) is a $\mathscr{T}$-pair if, roughly speaking, B and D are translation invariant [see Schäffer, Math. Ann. **137** (1959), 209–262; **138** (1959), 141–144; MR **21** #287, 6529].

In what follows, suppose that (B, D) is an admissible $\mathscr{T}$-pair (whence D contains non-trivial solutions). Let X_{0D} be the linear manifold of initial points $x(0)$ of solutions of (I) which belong to D, and suppose that X_{0D} is closed (but its complement need not be closed) and $X_{0D} \neq \{0\}$. Then for every solution of (I): $x(0) \in X_{0D}$ implies, for $t \geqq t_0 \geqq 0$,

$$(3) \quad \begin{gathered} \int_t^{t+\Delta} \|x(s)\| \, ds \leqq M_0(\Delta) \int_{t_0}^{t_0+\Delta} \|x(s)\| \, ds, \\ \|\chi_{[t,t+\Delta]} x\|_D \leqq M_0(\Delta) \|\chi_{[t_0,t_0+\Delta]} x\|_D; \end{gathered}$$

and $x(0) \in' X_{0D}$, $\gamma[X_{0D}, x(0)] \geqq \gamma$, imply, for $t \geqq t_0 \geqq 0$,

$$(4) \quad \begin{gathered} \int_t^{t+\Delta} \|x(s)\| \, ds \geqq M_1(\Delta, \gamma) \int_{t_0}^{t_0+\Delta} \|x(s)\| \, ds, \\ \|\chi_{[t,t+\Delta]} x\|_D \geqq M_1(\Delta, \gamma) \|\chi_{[t_0,t_0+\Delta]} x\|_D. \end{gathered}$$

Here χ_I is the characteristic function of I and $(-1)^i M_i$ are non-increasing with $\Delta > 0$. If, in addition, (B, D) is not weaker than (L^1, L_0^∞), then the inequalities (3), (4) hold with their right-hand sides multiplied by $e^{-\nu_0(t-t_0)}, e^{\nu_1(t-t_0)}$, respectively. Moreover, if D is also stronger than L^∞, then every solution of (I) which belongs to D satisfies, for $t \geqq t_0 \geqq 0$,

$$\|x(t)\| \leqq R(t_0) \|x(t_0)\| e^{-\nu(t-t_0)},$$

where $R(t_0)$ can not be taken independent of t_0 in general. Using these results, the authors then prove the following theorems on dichotomies.

If (B, D) is stronger than $(\Theta_\tau L^1, L^\infty)$ for some $\tau \geq 0$, where $\Theta_\tau L^1$ denotes the subspace $\{\chi_{(\tau,\infty)}u : u \in L^1\}$, then X_{0D} induces a dichotomy. Conversely, if a closed linear subspace induces a dichotomy then (L^1, L^∞) is admissible. If (B, D) is not stronger than $(\Theta_\tau L^1, L^\infty)$ then X_{0D} need not induce a dichotomy unless some other condition such as (*) is imposed, in which case X_{0D} always induces a dichotomy.

If (B, D) is not weaker than (L^1, L_0^∞) and if there exists a subspace Y which induces a dichotomy, then X_{0D} induces an exponential dichotomy. This is a necessary condition as well provided (*) holds.

As in the previous parts, the authors again give numerous examples to illustrate the interplay between the various hypotheses and to show that in almost all cases the results obtained are best possible. They also point out that, given a $\mathscr{T}$-pair (B, D), there does not in general exist an equation (II) with respect to which it is admissible.

H. A. Antosiewicz (Los Angeles, Calif.)

8182:

Schäffer, Juan Jorge. Linear differential equations and functional analysis. V. Math. Ann. **140** (1960), 308–321.

This sequel to the paper reviewed above is mainly concerned with further properties of admissible $\mathscr{T}$-pairs (B, D). [See the preceding review for notations and terminology.] The author proves that if such a pair is admissible there exists a weakest B_0 [a strongest D_0] such that (B_0, D) [(B, D_0)] is also an admissible $\mathscr{T}$-pair. He shows, further, that a complete description of the set of admissible $\mathscr{T}$-pairs can be given provided there exists a dichotomy.

For example, suppose a closed linear subspace Y induces a dichotomy, (B, D) is an admissible $\mathscr{T}$-pair, and X_{0D} is closed. Then X_{0D} also induces a dichotomy and $X_{0D} = X_0$ or $X_{0D} = X_{00}$ according as D is or is not weaker than L^∞. Here X_0 [X_{00}] denotes the linear manifold of initial points $x(0)$ of bounded solutions $x(t)$ of (I) [such that $\|x(t)\| \to 0$ as $t \to \infty$]. In particular, if X_{00} induces a dichotomy and X_0 is closed [is not closed], then a $\mathscr{T}$-pair (B, D) is admissible and X_{0D} is closed if and only if (B, D) is weaker than (L^1, L_0^∞) [and not weaker than (L^1, L^∞)]. If Y induces an exponential dichotomy and B is given, there exists a D_0 stronger than L^∞ such that (B, D) is admissible if and only if D is weaker than D_0.

When X is finite dimensional and the matrix $A(t)$ in (I) satisfies (*) it follows that the set of admissible $\mathscr{T}$-pairs is either empty or coincides with that for the equation $x' = f(t)$ (dichotomy) or $x' + x = f(t)$, $x' - x = f(t)$ (exponential dichotomy).

H. A. Antosiewicz (Los Angeles, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 8142, 8143, 8144, 8156, 8158, B8664, B8665.

8183:

Langer, Rudolph E. (Editor). ★Partial differential equations and continuum mechanics. Proceedings of an international conference conducted by the Mathematics Research Center at the University of Wisconsin, Madison, June 7–15, 1960. The University of Wisconsin Press, Madison, Wis., 1961. xv + 397 pp. $5.00.

In addition to abstracts of contributed papers, this volume contains the hour lectures of the following invited speakers (each will be reviewed individually): C. Müller, S. Agmon, Å. Pleijel, A. Weinstein, G. Fichera, R. Stonely, B. R. Seth, J. Kampé de Fériet, J. Leray, J. Moser, H. Lewy, I. Imai, F. G. Tricomi, T. M. Cherry, L. Hörmander, C. B. Morrey, Jr., D. P. Riabouchinsky, J. M. Burgers, K. Nickel.

8184:

Žukov, A. I. A limit theorem for difference operators. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 129–136. (Russian)

Consider the parabolic equation

$$(1) \qquad \frac{\partial u}{\partial t} = a_0 u + \cdots + a_n \frac{\partial^n u}{\partial x^n}$$

(real constant coefficients) and the initial value problem determined by giving $u^{(0)}(x) = u(t_0, x)$ for all x at some time $t = t_0$. If u is the solution and if for a fixed time step τ we write $u^{(1)}(x) = u(t_0 + \tau, x)$ then $u^{(1)} = Fu^{(0)}$ defines a linear operator F. Suppose we are also given some difference procedure

$$(2) \qquad u^*(x) = Gu^{(0)}(x) = \sum_m b_m u^{(0)}(x + mh)$$

approximating (1). (The b's are real; m runs over a fixed set of integers; h is a fixed Δx.) The result of following (2) through n time steps is then $G^n u^{(0)} = u^{*(n)}$ while $F^n u^{(0)} = u^{(n)}$ is the value $u(t_0 + n\tau, x)$ of the solution after the same time lapse, and the asymptotic accuracy of (2) as $t \to \infty$ can be studied in terms of the relation of G^n to F^n as $n \to \infty$.

Let $\tilde{f}$ denote the Fourier transform of a function f. (All transforms and convolutions are understood in the sense of Gel'fand and Šilov [Uspehi Mat. Nauk **8** (1953), no. 6 (58), 3–54; MR **15**, 867].) Then $\tilde{u}^{(1)}(y) = \phi(y)\tilde{u}^{(0)}(y)$ and $\tilde{u}^*(y) = \psi(y)\tilde{u}^{(0)}(y)$, where $\log \phi(y) = \sum_{k=0}^n (-1)^k a_k (iy)^k$ while $\psi(y) = \sum_m b_m e^{-imhy}$ has a power series expansion $\psi(y) = \sum_{k=0}^\infty \alpha_k / k! (iy)^k$ and, assuming $\alpha_0 = \sum_m b_m > 0$ (this amounts to the reasonable restriction that G should preserve the sign of constants), so does $\log \psi(y) = \sum_{k=0}^\infty \chi_k / k! (iy)^k$. The author regards ψ as a generalization of the characteristic function of a probability measure; the α's [χ's] are accordingly called the moments [resp. semi-invariants] of G. The index p of G is then defined as the order of the first of its moments [semi-invariants] that fails to agree with the corresponding moment [semi-invariant] of F, so that for $k < p$, $\chi_k = (-1)^k k! \tau a_k$ while $\chi_p \neq (-1)^p p! \tau a_p$. (The index is also the largest integer p such that $G = F$ for all real polynomials of degree $< p$.)

Now, theorem: Let G have degree $p \geq 2$ and let $\beta = \chi_p - (-1)^p p! \tau a_p$ $(\beta \neq 0)$ and $\beta = \varepsilon|\beta|$ $(\varepsilon = \pm 1)$. Let also $\sigma = (n|\beta|)^{1/p}$ and write

$$f_p(x) = (1/2\pi) \int_{-\infty}^{+\infty} \exp(-ixy) \exp(\varepsilon (iy)^p / p!)\, dy.$$

Then $u^{*(n)}(\sigma x) \to f_p(x) * u^{(n)}(\sigma x)$ as $n \to \infty$ (convergence in the sense of generalized functions). Note that except for the sign ε the functions f_p are independent of F and G. For p even, $\varepsilon i^p = -1$ is assumed as a reasonable stability restriction; then f_2 is just the normal probability density. Graphs of f_3 (an Airy function) and f_4 are given. The author shows how a proof of this theorem can be modeled on that of the central limit theorem, of which it provides, in

a sense, a generalization. Some generalizations are indicated and some special cases given detailed consideration.

It is important to note that the principal intent of this paper is heuristic; the main interest is not in the above theorem as such but in the point of view it suggests. The claim is made that other aspects of classical probability theory admit analogous generalizations bearing on the structure of parabolic and hyperbolic difference operators.

A. Brown (Houston, Tex.)

8185:

Kuznecov, N. N.; Roždestvenskiĭ, B. L. Construction of the generalized solution of the Cauchy problem for a quasi-linear equation. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 211–215. (Russian)

Generalized solutions of the Cauchy problem for $u_t+\phi_x(u,t,x)=0$ in $t\geqq 0$ are considered. $\Phi(t,x)=\int_{(0,0)}^{(t,x)} u\,dx-\phi(u,t,x)dx$ is a continuous solution of $\Phi_t+\phi(\Phi_x,t,x)=0$. Let $X(t,x_0)$, $U(t,x_0)$, $\Phi(t,x_0)$ represent the solution of the characteristic strip equations

$$\frac{dx}{dt}=\phi_u(u,t,x),\quad \frac{dU}{dt}=-\phi_x,\quad \frac{d\Phi}{dt}=U\phi_u'-\phi$$

through $t=0$, $x=x_0$, $U=u(0,x_0)$, $\Phi=\Phi(0,x_0)$. The authors consider the graph of $X(t,x_0)$, $\Phi(t,x_0)$ for fixed t and variable x_0 in an (X,Φ)-plane. Every generalized solution is obtained by piecing together segments of this graph so that Φ becomes a one-valued continuous function of X.

P. Ungar (New York)

8186:

Kuznecov, N. N.; Roždestvenskiĭ, B. L. Existence and uniqueness of the generalized solution of the Cauchy problem for a non-homogeneous law of conservation. Dokl. Akad. Nauk SSSR **126** (1959), 486–489. (Russian)

In an earlier paper B. L. Roždestvenskiĭ [same Dokl. **122** (1958), 551–554; MR **21** #185] has studied the uniqueness of the generalized solution of the Cauchy problem for

$$(1)\quad \partial u_i/\partial t+\partial\varphi_i(u,t,x)/\partial x=0,\quad u=\{u_1,u_2,\cdots,u_n\},$$

for the case $n=1$ by considering an equivalent problem in the "potential" Φ_1 of u_1, namely that of finding continuous solutions of the Cauchy problem for a nonlinear equation satisfied by Φ_1. Using the notion of potential introduced in the paper mentioned above for a homogeneous law of conservation [B. L. Roždestvenskiĭ, ibid. **115** (1957), 454–457; MR **20** #175], the authors stated corresponding to this law, in a previous paper [#8185], the question of existence and uniqueness of the generalized solution of the Cauchy problem for (1), $n=1$. Applying the method of successive approximation, the authors prove here the equivalence of the existence and uniqueness of the generalized solution $u(t,x)$ of the Cauchy problem for the equation

$$\partial u/\partial t+\partial\varphi(u,t,x)/\partial x=\psi(u,t,x)$$

(a non-homogeneous law of conservation), and that of the uniqueness and existence of the continuous fragmentary-differentiable solution of the equation

$$\partial\Phi/\partial t+\varphi(\partial\Phi/\partial x,t,x)=\int_0^x \psi(\partial\Phi(t,\xi)/\partial\xi,t,\xi)\,d\xi,$$

with the condition

$$\Phi(0,x)=\Phi_0(x)=\int_0^x u_0(\xi)\,d\xi,$$

where

$$\Phi(t,x)=\int_{(0,0)}^{(t,x)} u(t,x)\,dx-[\varphi(u,t,x)-F(t,x)]\,dt$$

is called the potential of the generalized solution of the initial Cauchy problem and

$$F(t,x)=\int_0^x \psi(u(t,\xi),t,\xi)\,d\xi.$$

D. Mangeron (Iaşi)

8187:

Oleĭnik, O. A. Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 165–170. (Russian)

Let $\varphi(u,t,x)$ have continuous second derivatives. A generalized solution of

$$u_t+(\varphi(u,t,x))_x=0$$

is a piecewise continuous function u such that $\int(u\,dx-\varphi(u,x,t)dt)=0$ along closed contours.

Set $u(t,x+0)=u_+$, $u(t,x-0)=u_-$. "Condition E" is: at every point x,t

$$(u_+-u_-)\varphi(w)-(w-u_-)\varphi(u_+)-(u_+-w)\varphi(u_-)\geqq 0 \quad \text{for } w\in[u_+,u_-].$$

Generalized solutions of Cauchy's problem satisfying condition E are constructed in articles by I. M. Gel'fand [same Uspehi, 87–158; MR **22** #1736] and the author [#8205]. Here condition E is shown to imply uniqueness and continuous dependence. The identity

$$\int_\Gamma (u-v)\,dx-(\varphi(u)-\varphi(v))\,dt=0,$$

valid for any two general solutions u, v and any closed contour Γ, is applied to a contour consisting of an arbitrary segment $t=$const., two solutions of

$$(u-v)\,dx=(\varphi(u)-\varphi(v))\,dt$$

and part of the x-axis.

P. Ungar (New York)

8188:

Roždestvenskiĭ, B. L. Conservativeness of systems of quasi-linear equations. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 217–218. (Russian)

A conservation law is an equation of the form

$$\frac{\partial\phi(u,t,x)}{\partial t}+\frac{\partial\psi(u,t,x)}{\partial x}=f(u,t,x),$$

where u is a vector and ϕ, ψ, f are scalars.

The system

$$\frac{\partial u_i}{\partial t}+u_{i+1}\frac{\partial u_i}{\partial x}=0\quad(i=1,2,3;\ u_4=u_1)$$

is given as an example of a totally non-conservative system. It does not have even a single conservation law as a consequence.

P. Ungar (New York)

8189:

Saito, Tomoyuki. The perturbation method due to the small change in the shape of the boundary. J. Phys. Soc. Japan **15** (1960), 2069–2080.

This paper deals with the formal theory of "boundary perturbation" for partial differential equations. Consider an equation such as $L\Psi(x)=-h(x)$ in a "perturbed domain" $G\in R^N$, which can be obtained from the "unperturbed domain" G_0 by a transformation of the form $\bar{x}\to x=\bar{x}+\varepsilon\varphi(\bar{x})+\cdots$, where ε is a small parameter. The covariant transformation of a scalar function $(\bar{u}(\bar{x})=u(x))$ is then given by $\bar{u}=u+\varepsilon Du+\cdots$ with $D=\varphi\cdot\text{grad}$, and the transformed operator $(\bar{L}\bar{u}=\overline{Lu})$ has the form $\bar{L}=L+\varepsilon(DL-LD)+\cdots$. Thus the original equation is transformed into $\bar{L}\bar{\Psi}=-\bar{h}$ in G_0, the solution of which is then obtained as a formal series in ε by the usual perturbation theory. The following related problems are discussed in detail and a number of useful formulas are deduced: the transformation formulas of tensor functions; the explicit formulas for $\bar{L}$ for such important examples as the Laplacian, D'Alembertian and heat equation; the transformation formulas for various boundary conditions; the eigenvalue problem, especially the perturbation series for the eigenvalues and the eigenfunctions. However, the paper is subject to the limitations that the treatment is quite formal, without any consideration of convergence or error estimate, and that some of the formulas are given only up to the first order in ε. *T. Kato* (Tokyo)

8190:

Meyer, A. G. Fehlerabschätzungen für die Ableitungen bei Randwertaufgaben mit linearer elliptischer Differentialgleichung. Arch. Rational Mech. Anal. **6**, 422–439 (1960).

In the closure $G+\Gamma$ of the n-dimensional bounded region G let $\{a_{jk}(P)\}$ be a positive definite matrix and $c(P)$ a non-negative function. Let

$$L[u]\equiv\sum_{j,k=1}^{n}a_{jk}(P)\frac{\partial^2u}{\partial x_j\partial x_k}-\sum_{j=1}^{n}b_j(P)\frac{\partial u}{\partial x_j}+cu(P).$$

The author derives explicit bounds for the derivatives of solutions of differential equations $L[u]=F(P)$ with a sufficiently smooth given function $F(P)$. These bounds are in terms of u itself. In general the inequalities apply only to points of G, but in certain cases, important in fluid dynamics, the tangential derivatives on Γ can also be estimated. The method follows the ideas of J. Schauder [Math. Z. **38** (1934), 257–282]. *W. Wasow* (Madison, Wis.)

8191:

Harris, W. A., Jr. Singular perturbations of two-point boundary problems for systems of ordinary differential equations. Arch. Rational Mech. Anal. **5**, 212–225 (1960).

In this paper the author considers the relation of solutions of

$$\varepsilon^{h_j}\frac{dx_j}{dt}=\sum_{k=1}^{p}A_{jk}(t,\varepsilon)x_k(t,\varepsilon)\quad(j=1,2,\cdots,p;\ h_1=0),$$

subject to two point boundary conditions $R(\varepsilon)x(a,\varepsilon)+S(\varepsilon)x(b,\varepsilon)=c(\varepsilon)$; to the degenerate problem for $\varepsilon=0$. Here x_j is a vector of dimension n_j, x is thus a vector of dimension $u=\sum_{j=1}^{p}n_j$, and R and S are square matrics of order n. Several transformations are employed to reduce the system to a canonical form for which the asymptotic form of the solution is known. Under assumptions which are too interrelated to describe here, it is shown that the solution of the perturbed problem approaches, as $\varepsilon\to0+$, a solution of the degenerate system which satisfies an appropriate number of the degenerate boundary conditions (not all of them). *R. R. D. Kemp* (Kingston, Ont.)

8192:

Sangren, Ward C. Interfaces in two dimensions. SIAM Rev. **2** (1960), 192–199.

The steady state temperature distribution for heat conduction in two rectangular coordinates satisfies the elliptic partial differential equation $T_{xx}+T_{yy}=0$. The solution to this equation in the rectangular region $0\leq x\leq a$, $0\leq y\leq b$ when there are interfaces at $x=x_1$ and $y=y_1$, $0<x_1<a$, $0<y_1<b$, may be obtained by separation of variables $(T(x,y)=X(x)Y(y))$ providing (1) the thermal conductivities of each subregion satisfy the condition $K_1K_4=K_2K_3$ (the subregion 1 is $0<x<x_1$, $0<y<y_1$; subregion 2 is $x_1<x<a$, $0<y<y_1$; subregion 3 is $0<x<x_1$, $y_1<y<b$; and subregion 4 is $x_1<x<a$, $y_1<y<b$) and (2) the boundary conditions are that $T=0$ on all boundaries except for $T(x,0^+)=f(x)$. The interface conditions for the temperature and its gradient are also satisfied.

The method of solution is extended to the classical membrane problem where the displacement satisfies $V_{xx}+V_{yy}+B^2V=0$ and $V=0$ on all boundaries.

It is indicated how the method may be applied to the heat conduction problem when there are I parallel interfaces perpendicular to the x-axis and J parallel interfaces perpendicular to the y-axis.

{The solutions obtained should be useful in error analyses of difference solutions to problems involving continuous non-homogeneous materials.} *G. W. Evans, II* (Menlo Park, Calif.)

8193:

Friedrichs, K. O. Boundary problems of linear differential equations independent of type. Boundary problems in differential equations, pp. 3–9. Univ. of Wisconsin Press, Madison, 1960.

The general theory of symmetric positive linear differential equations is outlined. Writing a system of linear equations for a vector u as $Ku=f$ where

$$K=\sum_{\rho}\left(\alpha^\rho\frac{\partial}{\partial x_\rho}+\frac{\partial}{\partial x_\rho}\alpha^\rho\right)+\kappa\text{ with }\alpha^\rho,\ \kappa\text{ matrices,}$$

the conditions that it be symmetric positive are (1) α^ρ is symmetric, (2) $\kappa+\kappa'$ is positive definite (κ' is the transpose of κ). The importance of such a formulation is that it is independent of type. The existence of weak solutions under appropriate boundary conditions was given in Comm. Pure Appl. Math. **11** (1958), 333–418 [MR **20** #7147]. The outline includes a summary of the main results of the above, later results relating weak, strong and differentiable solutions and a discussion of the problem of finding numerical solutions. Tricomi-type equations are particularly discussed. *C. S. Morawetz* (New York)

8194:

Šišmarev, I. A. A priori estimation of solutions to the Dirichlet problem for an elliptic operator with discontinuous coefficients. Dokl. Akad. Nauk SSSR **131**

(1960), 269–272 (Russian); translated as Soviet Math. Dokl. 1, 249–252.

Let g be an open N-dimensional domain bounded by Γ_2 and partitioned into subdomains g_1 and g_2 by a closed surface Γ_1. The author establishes the estimate

$$\max_{x\in(g+\Gamma_2)} |u(x)| \leq \max_{x\in\Gamma_1} |k(x)|$$

for the solution to the Dirichlet problem: $L_1u=0$ in g_1, $L_2u=0$ in g_2, $[u]|_{\Gamma_1}=0$, $[\partial u/\partial v]|_{\Gamma_1}=k$, $u|_{\Gamma_2}=0$, where L_i is a linear differential operator of elliptic type defined in g_i with $L_1 \not\equiv L_2$, and k is a given function defined in Γ_1.

R. N. Goss (San Diego, Calif.)

8195:

Jones, E. E. A circle theorem associated with the Poisson differential equation. J. Math. Mech. 9 (1960), 563–572.

Let S_e $[S_i]$ be the region exterior [interior] to a circle C in the plane with center at the origin. The problem considered is to determine two functions $\phi_e(x, y)$ and $\phi_i(x, y)$ having prescribed singularities and satisfying

$$\nabla^2\phi_e = F_e(x, y) \quad \text{in } S_e,$$
$$\nabla^2\phi_i = F_i(x, y) \quad \text{in } S_i,$$

where F_e and F_i are given functions in S_e and S_i, respectively, subject to the boundary conditions

$$\phi_e = \phi_i, \quad \alpha_e \frac{\partial\phi_e}{\partial r}+\beta_e\phi_e = \alpha_i \frac{\partial\phi_i}{\partial r}+\beta_i\phi_i+\psi$$

on C, where α_e, β_e, α_i, β_i are given constants and ψ a given function on C. Under suitable regularity conditions, a method for constructing a solution is given, and a uniqueness theorem is proved. This generalizes results of Power and Jackson [Appl. Sci. Res. B 6 (1957), 456–460; MR 19, 848].

M. Schechter (New York)

8196:

Jones, Evan E. Solution of the Poisson equation related to a boundary-value problem for the sphere. Z. Angew. Math. Phys. 11 (1960), 265–272. (German summary)

Consider the equations (1) $\Delta\varphi_e=f_e$, $\Delta\varphi_i=f_i$ in the exterior and interior of a unit sphere S, respectively, and the boundary conditions on S, (2) $\varphi_e=\varphi_i$, $\alpha_e\partial\varphi_e/\partial r+\beta_e\varphi_e=\alpha_i\partial\varphi_i/\partial r+\beta_i\varphi_i+\psi$. f_e and f_i are assumed to be continuously differentiable, the α's and β's are real constants, and ψ is a continuous function on S. By methods of potential theory the author proves the existence of a solution of the problem (1), (2).

A. Friedman (Minneapolis, Minn.)

8197:

Meyer, A. G. Schranken für die Lösungen von Randwertaufgaben mit elliptischer Differentialgleichung. Arch. Rational Mech. Anal. 6, 277–298 (1960).

In an open bounded region G with boundary Γ in Euclidean n-space consider a second order elliptic differential expression $E[u]$, not necessarily linear, for a function $u=u(x_1, x_2, \cdots, x_n)$. On Γ there is defined a first order differential expression $S[u]$. Under rather mild conditions the following extension of the maximum principle is proved: If the functions $u^{(1)}$ and $u^{(2)}$ do not differ by a constant then $E[u^{(1)}]\leq E[u^{(2)}]$ in G and $S[u^{(1)}]\leq S[u^{(2)}]$ on Γ imply $u^{(1)}\leq u^{(2)}$ in $G+\Gamma$. The assumptions are too long to be stated here. They permit the presence of "corners" and "edges" on Γ, but not of inward "cusps". In the special case that $S[u]$ is linear it must be of the form $A_1u-A_2u_\mu$, where A_1, A_2 are bounded non-negative functions on Γ and u_μ an inward directed derivative. An extension to an unbounded region G is also proved. It is further shown how this theorem can be used to include the solutions of boundary value problems of the form $E[u]=g$ in G, $S[u]=\gamma$ on Γ, with $S[u]$ linear, between explicit bounds. This requires, roughly speaking, the construction of approximate solutions q for which $E[q]$ and $S[q]$ have preassigned signs.

W. Wasow (Madison, Wis.)

8198:

Payne, L. E.; Weinberger, H. F. An optimal Poincaré inequality for convex domains. Arch. Rational Mech. Anal. 5, 286–292 (1960).

The authors obtain a new lower bound to the lowest non-zero eigenvalue of $\nabla^2v+\mu v=0$ in a convex domain G, with $\partial v/\partial n=0$ on the boundary. If this eigenvalue is called μ_2, then $\mu_2\geq\pi^2D^{-2}$, where D is the diameter of G.

I. Stakgold (Evanston, Ill.)

8199:

Roškulec, Maršel N. [Roşculeţ, Marcel N.]. Infinite algebras related to Laplace's equation. Rev. Math. Pures Appl. 3 (1958), 231–264. (Russian)

This paper continues the study of the algebra A of infinite order (associated to the Laplacian equation) which was introduced earlier by the author [Com. Acad. R. P. Romîne 5 (1955), 1245–1252; MR 17, 1071]. Firstly there is given an integral expression of the elementary solution and of a harmonic polynomial. Secondly the author obtains a Cauchy formula for the monogenic functions on A and also integral expressions for the coefficients of the development $f(w)=\sum a_m(w-w_0)^m$, w_0, $w \in A$.

An extension of these results to a partial differential equation with constant coefficients is also indicated.

C. Foiaş (Bucharest)

8200:

Meyers, Norman; Serrin, James. The exterior Dirichlet problem for second order elliptic partial differential equations. J. Math. Mech. 9 (1960), 513–538.

This paper contains a careful and lucid analysis of the behavior of solutions of second order elliptic equations, which are defined in a neighborhood $\mathcal{E}$ of infinity in $n\geq2$ dimensions, and which assume prescribed boundary data on an inner boundary $F\mathcal{E}$. To fix the ideas, consider an elliptic equation of the form (*) $a_{ik}(\partial^2u/\partial x_i\partial x_k)=0$, with Dirichlet data $\mu=\varphi(x)$ on $F\mathcal{E}$, and one of the following types of boundary condition at infinity: (I) $\mu(x)$ tends to a prescribed constant l as $x\to\infty$; (II) $\mu(x)$ is bounded in $\mathcal{E}$. As is known, if $a_{ik}=\delta_{ik}$, then problem I is well-set for $n\geq3$ in the sense that there is always a unique solution. On the other hand, if $a_{ik}=\delta_{ik}$ and $n=2$, then II is correctly set, and I has in general no solution. In this latter case the solution tends to a limit which is determined by the inner data $\varphi(x)$ on $F\mathcal{E}$. The authors show that this behavior generalizes to equations (*) in a surprising way. Let $\{\lambda_i\}$ be the eigenvalues of the matrix (a_{ik}) and let $Q=a_{ik}x_ix_k/x_i^2$. Let $A(x)=\sum\lambda_i/Q$. By ellipticity, $1<A(x)<\infty$. Suppose that, in $\mathcal{E}$, $A(x)\geq2+\varepsilon(r)$, where $\varepsilon(r)$

is such that the function $\Delta(t) = \exp\{-\int^t \varepsilon(r)s^{-1}\,ds\}$ satisfies $d[\Delta] = \int^\infty \Delta(t)t^{-1}\,dt < \infty$. Then problem I is well-set for (*). Conversely, if $A \leqq 2 + \varepsilon(r)$ and $\varepsilon(r)$ is such that $d[\Delta] = \infty$, then problem II is well-set. These results are sharp, as is shown by a simple example.

As an application, one obtains that if $(a_{ik}) \to (a_{ik}^0)$ at infinity, and if (a_{ik}^0) has at least three positive eigenvalues, then problem I is well-set. Further, the authors observe that if $n=2$, problem I can be well-set even for some equations for which $a_{ik} \to \delta_{ik}$ at infinity. Thus, for $n=2$ the determination of the properly set problem for an equation requires considerable knowledge of the asymptotic behavior of the coefficients, while for $n > 3$ problem I remains well-set even for equations with pronounced parabolic degeneration at infinity.

The paper contains various further results, including extensions to more general equations and to other boundary conditions on $F\mathscr{E}$. The demonstrations are elementary, being based for the most part on the explicit construction of barriers at infinity. *R. Finn* (Stanford, Calif.)

8201:

Kreyszig, Erwin. Zur Behandlung elliptischer partieller Differentialgleichungen mit funktionentheoretischen Methoden. Z. Angew. Math. Mech. **40** (1960), 334–342. (English and Russian summaries)

This article gives a short and very readable introduction to Bergman's integral operator method and some of its applications in the treatment of partial differential equations of elliptic type in two and three variables.

Z. Nehari (Pittsburgh, Pa.)

8202:

Kamke, Erich. Über die erste Randwertaufgabe bei der Laplace- und der Wärmeleitungs-Differentialgleichung. Jber. Deutsch. Math. Verein **62** (1959), Abt. 1, 1–33.

L'autore prende le mosse dall'osservare che la risoluzione del cosiddetto primo problema al contorno relativo all'equazione differenziale di Laplace

$$(1)\quad \Delta_n u = 0, \quad u = u(x_1, x_2, \cdots, x_n), \quad \Delta_n u = \sum_{\nu=1}^{n} \frac{\partial^2 u}{\partial x_\nu^2},$$

per dominii generali, si basa sulla risoluzione corrispondente per una sfera (vedasi il metodo di Perron [cf. Math. Z. **18** (1923), 42–54] delle funzioni superiori ed inferiori), come pure quest'altra, messa in evidenza dallo W. Sternberg per $n=2$ [cf. Math. Ann. **101** (1929), 394–398], e che analogamente il problema al contorno relativo ad una certa equazione parabolica e precisamente quella della propagazione del calore

$$(2)\quad u_{x_n} = \Delta_{n-1}u, \quad u = u(x_1, x_2, \cdots, x_n), \qquad \Delta_{n-1}u = \sum_{\nu=1}^{n-1} \frac{\partial^2 u}{\partial x_\nu^2}$$

si basa sulla risoluzione corrispondente essa pure ad un dominio speciale (per $n=2$: un trapezio). L'autore elabora in questa sua memoria un metodo unitario che permette di far risalire da domini speciali a domini più generali per l'insieme dei problemi al contorno spettanti in comune ad ambedue le equazioni alle derivate parziali citate.

Nel § 1 si introduce per ambedue i problemi una topologia T più generale di quella di R_n ordinariamente utilizzata, e cioè: (A_1) T è uno spazio di Hausdorff; (A_2) esistono intorni $V(x)$ arbitrariamente piccoli: ciò vuol dire che per ogni $U(x)$ vi è un $V(x) \subseteqq U(x)$; (A_3) per ogni $V(x)$ vi è un $U(y) \subseteqq V(x)$; (A_4) ogni insieme $\mathfrak{M}$ limitato che contiene almeno due punti ha la frontiera $R(\mathfrak{M})$ che contiene anch'essa almeno due punti. Dopo aver introdotto nei §§ 2 e 3 le nozioni di dominii elementari, di operatori, di sottofunzioni, di soprafunzioni e di funzioni totali (Vollfunktionen), che seguono como metodo i procedimenti già utilizzati dal Perron [loc. cit.], l'autore riesce a presentare il problema al contorno, § 4, e la sua risoluzione, § 5, in una forma ancora più generale di quella richiesta per la considerazione unitaria delle equazioni (1) e (2). Il recensore considera necessario sottolineare che i risultati conseguiti senza molto spreco sono atti ad essere utilizzati anche in altri domini. L'ultimo § 6, pur collegandosi ai risultati importanti ottenuti dal I. Petrovskiĭ [cf. Compositio Math. **1** (1935), 383–419], è dedicato ad alcuni problemi speciali concernenti l'equazione della propagazione del calore. Se ne deduce fra l'altro che il primo problema al contorno è risolubile per ogni sfera. *D. Mangeron* (Iaşi)

8203:

Zemach, C.; Odeh, F. Uniqueness of radiative solutions to the Schroedinger wave equation. Arch. Rational Mech. Anal. **5**, 226–237 (1960).

Consider the Schroedinger equation

$$(*)\qquad [\nabla^2 + k^2 - V(\mathbf{r})]\psi = 0,$$

in a domain $D \subset R^3$ exterior to a bounded surface. It is assumed that $V(\mathbf{r})$ is real and bounded at infinity, $I(\mathbf{r}) = \int_D |\mathbf{r}-\mathbf{s}|^{-1}|V(\mathbf{s})|ds$ is finite and continuous in $\mathbf{r} \in D$ and $I(\mathbf{r}) \to 0$ for $r = |\mathbf{r}| \to \infty$. A solution of (1) satisfying the Sommerfeld radiation condition

$$(**)\qquad \lim_{R\to\infty} \int_{r=R} |\partial\psi/\partial r - ik\psi|^2\, dS = 0$$

is called a "wave function". The author proves the following theorems. (1) Any wave function is $O(r^{-1})$ at infinity. (2) Any wave function ψ such that $\lim_{R\to\infty} \int_{r=R} |\psi|^2 dS = 0$ is identically zero. (3) The solution of the Dirichlet or Neumann problem for (*) in D is unique if (**) is required. The proof of these theorems depends on an integral representation of the wave function in terms of the Green function $G(\mathbf{r}, \mathbf{s}) = -\exp(ik|\mathbf{r}-\mathbf{s}|)/4\pi|\mathbf{r}-\mathbf{s}|$. It appears that there is a gap in the proof of (2) which is, however, corrected by the author elsewhere. In any case, (2) follows directly from a theorem of the reviewer [Comm. Pure Appl. Math. **12** (1959), 403–425; MR **21** #7349] under slightly different assumptions on $V(\mathbf{r})$, even without assuming that ψ satisfies the radiation condition (**).

T. Kato (Tokyo)

8204:

Egorov, Yu. V. Hyperbolic equations with discontinuous coefficients. Dokl. Akad. Nauk SSSR **134** (1960), 514–517 (Russian); translated as Soviet Math. Dokl. **1**, 1095–1098.

This paper is concerned with the proof of existence, uniqueness, and smoothness of solutions of the mixed problem for general linear hyperbolic equations of the second order with discontinuous coefficients. The author applies methods and results of O. A. Oleĭnik [same Dokl. **124** (1959), 1219–1222; MR **21** #1442], approximates the

coefficients by smooth functions, and applies a priori integral estimates and results of S. M. Nikol'skiĭ [Trudy Mat. Inst. Steklov. 38 (1951), 244–278; MR 14, 32]. After showing existence of generalized solutions under very general conditions, the author shows regularity of the solutions in case the coefficients have discontinuities of the first kind on smooth boundaries of a finite number of subdomains. In the second part of the paper the author applies similar considerations to first order hyperbolic systems, treating the Cauchy problem for these. [See also the review following.]

A. N. Milgram (Minneapolis, Minn.)

8205:

Oleĭnik, O. A. Construction of a generalized solution of the Cauchy problem for a quasi-linear equation of first order by the introduction of "vanishing viscosity". Uspehi Mat. Nauk **14** (1959), no. 2 (86), 159–164. (Russian)

As $\varepsilon \to 0$, the solutions of the equation

$$\varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2} = \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial \phi(u_\varepsilon, t, x)}{\partial x}, \quad u_\varepsilon(0, x) = u_0(x),$$

converge in the mean if u_0 is monotonic; if ϕ_{uu} is nowhere 0, u_0 need only be bounded measurable. The limit is a weak solution of

$$\frac{\partial u}{\partial t} + \frac{\partial \phi(u, t, x)}{\partial x} = 0,$$

which is the main object of interest. The proof is sketched. It uses the function

$$v_\varepsilon(t, x) = \int^{(t,x)} \left(\varepsilon \frac{\partial u_\varepsilon}{\partial x} - \phi(u_\varepsilon) \right) dt + u_\varepsilon\, dx,$$

which satisfies the more tractable equation

$$\varepsilon \frac{\partial^2 v_\varepsilon}{\partial x^2} = \frac{\partial v_\varepsilon}{\partial t} + \phi\left(\frac{\partial v_\varepsilon}{\partial x}\right).$$

The author gave more detail in Uspehi Mat. Nauk **12** (1957), no. 3 (75), 3–73 [MR **20** #1055].

P. Ungar (New York)

8206:

Walter, Wolfgang. Eindeutigkeitssätze für gewöhnliche, parabolische und hyperbolische Differentialgleichungen. Math. Z. **74** (1960), 191–208.

Strict solutions of initial value problems for $u_x = f(x, y, u, u_y, u_{yy})$ and $u_{xy} = f(x, y, u, u_x, u_y)$ are considered. The moduli of continuity needed in the theorems are characterized by requiring that solutions of certain differential inequalities exist. The theorems contain all previous results on the subject.

P. Ungar (New York)

8207:

Carroll, Robert. L'équation d'Euler-Poisson-Darboux et les distributions sousharmoniques. C. R. Acad. Sci. Paris **246** (1958), 2560–2562.

Soit $\mu_x(r)$ la moyenne sphérique: elle est la mesure sur R^n formée de la masse $+1$ répandue de façon homogène sur la sphère $|x| = r$. L'auteur remarque que $r^k \mu_x(r) \in \mathscr{E}_r^{k+2}(\mathscr{E}_x')$ pour $r \geq 0$ et tout entier $k \geq 0$, c'est-à-dire que $r^k \mu_x(r)$ est une fonction $(k+2)$-fois continuement différentiable à valeurs dans $\mathscr{E}_x'$. Considérons maintenant $\mu_x(r) * T$ pour $T \in \mathscr{D}_x'$. Elle est une distribution dépendant du paramètre r. Soit $L_r^{n-1} = (\partial^2/\partial r^2) + [(n-1)/r](\partial/\partial r)$. On a alors $L_r^{n-1}(\mu_x(r) * T) = \Delta * \mu_x(r) * T$, ce qui montre que $\mu_x(r) * T$ est une solution de l'équation d'Euler-Poisson-Darboux d'indice $n-1$. A partir de cette formulation, l'auteur montre que quelques théorèmes de Weinstein [Ann. Mat. Pura Appl. (4) **43** (1957), 325–340; MR **19**, 656] se traitent dans le cadre des distributions.

S. Mizohata (New York)

8208:

Kanel', Ya. I. Certain problems on equations in the theory of burning. Dokl. Akad. Nauk SSSR **136** (1961), 277–280 (Russian); translated as Soviet Math. Dokl. **2**, 48–51.

The behavior for $t \to \infty$ of solutions of $\partial u/\partial t - \partial^2 u/\partial x^2 = F(u)$ with the initial condition $u|_{t=0} = u_0(x)$ is investigated for various natural assumptions concerning the properties of $F(u)$.

F. Goodspeed (Quebec)

8209:

Čeremnyh, Yu. N. A theorem in the qualitative theory of parabolic equations. Dokl. Akad. Nauk SSSR **130** (1960), 33–36 (Russian); translated as Soviet Math. Dokl. **1**, 23–26.

The theorem provides an upper bound for

$$\max_{\Sigma^*} |u(T, x)|,$$

where $u(t, x)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} = a(t, x) \frac{\partial u}{\partial t} + b(t, x) \frac{\partial u}{\partial x} + c(t, x) u$$

in a certain closed subdomain of the unit square in the tx-plane, and Σ^* is a certain subinterval of the unit interval on the x-axis. The bound involves a domain constant and the number of essential components of the solution, i.e., subsets that have certain dispositions of limit points in the plane.

R. N. Goss (San Diego, Calif.)

8210:

Amerio, Luigi. Problema misto e quasi-periodicità per l'equazione delle onde non omogenea. Ann. Mat. Pura Appl. (4) **49** (1960), 393–417.

Soit Ω un domaine borné de l'espace éuclidien S_m; $x = (x_1, \cdots, x_m)$, et soit σ la frontière de Ω; $J \equiv -\infty < t < +\infty$. Supposons $a_{jk}(x)$, $a(x)$ réelles, vérifiant dans Ω les conditions

$$\sum_{j,k}^{1,\cdots,m} a_{jk}(x) \xi_j \xi_k \geq \nu \sum_1^m{}_j\, \xi_j^2, \quad \nu > 0,\ a(X) \geq 0.$$

La presque-périodicité par rapport à t des solutions nulles sur σ de l'équation des ondes homogène

$$(1) \qquad u_{tt}(x, t) = \sum \frac{\partial}{\partial x_j} \left(a_{jk}(x) \frac{\partial u}{\partial x_k} \right) - a(x) u(x, t)$$

a été déja démontrée de plusieurs façons par divers auteurs.

Dans ce mémoire on résout le problème mixte pour l'équation (1) d'une manière nouvelle, qui dans le cas d'une seule dimension est due à B. de Sz-Nagy [Bull. Soc. Math. France **75** (1947), 193–208; MR **10**, 269]. On obtient de nouveau la presque-périodicité par rapport à t des

solutions généralisées ainsi trouvées, et aussi l'unicité et la dépendance continue des données initiales et du second membre.

Soit $L^2(\Omega)$ l'espace des fonctions de carré sommable en Ω, et $D^\circ(\Omega)$ l'espace obtenu par complétion dans la norme

$$\|y(x)\|_{D^\circ} = \left\{\int_\Omega \left(\sum a_{jk}(x)\frac{\partial y}{\partial x_j}\frac{\partial y}{\partial x_k}+a(x)y^2\right)d\Omega\right\}^{1/2}$$

de l'ensemble D des fonctions de $C^1(\Omega)$ qui sont nulles près de σ. Soit $E=D^\circ\times L^2$. Considérons $f(x,t)$ une fonction de J à $L^2(\Omega)$, continue et presque-périodique. En généralisant un résultat de Bohr et Neugebauer [Nachr. Gesellschaft. Wiss. Göttingen. Math.-Phys. Kl. **1926**, 8–22], le rapporteur a démontré [J. Math. Mech. 8 (1959), 369–382; MR **21** #2812], la presque-périodicité des vecteurs de E, $\{u(x,t), u_t(x,t)\}$ formés à partir des solutions $u(x,t)$ de l'équation non-homogène

$$(2)\qquad u_{tt}(x,t) = \sum\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right)-a(x)u+f(x,t)$$

si on suppose de plus que ces vecteurs ont dans E la trajectoire relativement compacte. Pour avoir considéré des solutions fortes, le rapporteur a supposé en plus la dérivabilité forte de $f(x,t)$ de J à $L^2(\Omega)$.

Dans ce mémoire, l'auteur généralise ce résultat, avec une méthode différente, pour les solutions généralisées qu'il a construit; dans ce cas il n'est plus nécessaire de supposer la dérivabilité forte de $f(x,t)$. Il semble au rapporteur que la méthode utilisée ici est adaptable aussi au cas presque-périodique non-stationnaire: $a_{jk}=a_{jk}(x,t)$, presque-périodique en t.

{Remarques du rapporteur: (a) La considération des distributions vectorielles presque-périodiques de L. Schwartz permet une nouvelle généralisation du résultat; voir la Note du rapporteur, #8211.

(b) Une amélioration essentielle, qui permet de remplacer la condition de compacité relative du vecteur $\{u(x,t), u_t(x,t)\}$, par la condition nécessaire qu'il soit borné dans E, a été obtenue dans les Notes ultérieures de l'auteur (voir le réfferat suivant).}

S. Zaidman (Bucharest)

8211:

Zaidman, Samuel. Solutions presque périodiques des équations hyperboliques. C. R. Acad. Sci. Paris **250** (1960), 2112–2114.

The notation in this review is somewhat different from the author's own. Let

$$f(t)\sim\sum A_n e^{i\Lambda_n t}$$

be a (strongly) almost periodic function from R: $(-\infty<t<\infty)$ to a separable Hilbert space H, with continuous derivative $f'(t)$. Let L be a positive definite self-adjoint operator on H with pure point spectrum

$$0<\lambda_1\leqq\lambda_2\leqq\cdots\to\infty,$$

and let no value $\pm\lambda_m$ be contained in the closure of the sequence $\{\Lambda_n\}$. If $u(t)$ is a function from R to H which solves the equation

$$(*)\qquad \frac{d^2u}{dt^2} = -L^2u+f(t),$$

then $Lu(t)$ and $du(t)/dt$ are almost periodic. This is theorem 1 of this paper. Theorems 2 and 3 deal with the homogeneous and inhomogeneous case of (*) under the assumption that $f(t)$ and $u(t)$ are not point-functions but "distributions", and theorem 3 is so drafted that it reduces to theorem 1 under suitable smoothing ("regularization") of both sides of the equation. Finally theorem 4 refers to the telegrapher's equation

$$\frac{d^2u}{dt^2} = -L^2u-\gamma u+f$$

under the restriction $0<\gamma<2\lambda_1$.

S. Bochner (Princeton, N.J.)

8212:

Haimovici, M. Sur le prolongement des équations II-e ordre à une fonction inconnue de deux variables indépendantes et sur les transformations de ces équations. II. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 105–131. (Romanian. Russian and French summaries)

In a previous paper [same An. **1** (1955), 69–136; MR **18**, 486] the author considered the representation of an analytic second order partial differential equation (in Monge's notation for partial derivatives)

$$F(x_1, x_2, z, p_1, p_2, p_{11}, p_{12}, p_{22}) = 0,$$

by means of a Pfaffian system $\theta_1=0, \cdots, \theta_N=0$; and also studied differential relations between the solution manifolds of one such partial differential equation (and its corresponding Pfaffian system) and another partial differential equation of the same kind

$$G(\xi_1, \xi_2, \zeta, \pi_1, \pi_2, \pi_{11}, \pi_{12}, \pi_{22}) = 0,$$

(and its corresponding Pfaffian system $X_1=0, \cdots, X_M=0$). In the present paper finite relations are investigated.

J. B. Diaz (College Park, Md.)

8213:

Gårding, Lars. Some trends and problems in linear partial differential equations. Proc. Internat. Congress Math. 1958, pp. 87–102. Cambridge Univ. Press, New York, 1960.

This is a summary of the state of knowledge in linear partial differential equations as of August 1958 with observations on recent trends in the subject. The topics covered include constant coefficient operators, L^2 inequalities, hypoellipticity, Cauchy's problem, unique continuation, double differential operators, regularity of solutions of elliptic equations, boundary problems, and spectral theory.

M. Schechter (New York)

8214:

Gording, L. [Gårding, L.]. Some trends and problems in linear partial differential equations. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 137–152. (Russian)

Translation of the paper reviewed above.

8215:

Jaenicke, Joachim. Lösungen gegebener Charakteristik linearer Randwertprobleme elliptischer Differentialgleichungssysteme. Math. Nachr. **21** (1960), 223–232.

The author continues his research on boundary value problems of the form

$$(\mathrm{A})\qquad \alpha(s)u+\beta(s)v = f(s)$$

for elliptic systems of differential equations of the form

$$\text{(B)} \qquad \begin{aligned} u_x - v_y &= au + bv + c, \\ u_y + v_x &= \bar{a}u + \bar{b}v + \bar{c}. \end{aligned}$$

It is assumed that (x, y) ranges over a closed region $\mathfrak{F}$ with boundary $\mathfrak{R}$ of class $C_\mu{}^1$ $(0 < \mu < 1)$ and the coefficients and f are Hölder continuous $(\in C_\mu{}^0)$ on $\mathfrak{F}$ and α and $\beta \in C_\mu{}^1$ on $\mathfrak{R}$ with $\alpha^2 + \beta^2 > 0$. We define the characteristics n and k as the variations of the arguments of $\beta(s) + i\alpha(s)$ and $v(s) + iu(s)$, respectively, around $\mathfrak{R}$. The author proves the following results. (i) The system (A, B) with $c \equiv \bar{c} \equiv 0$ always has solutions with $k = n$; if $n = -\nu$, ν zeros may be prescribed in $\mathfrak{F} - \mathfrak{R}$. (ii) The general system has continuous solutions with $k \neq n$ and without zeros on $\mathfrak{R}$ only if $f(s)$ has at least $2|k-n|$ zeros with changes of sign. (iii) If $f(s)$ satisfies this condition the system (A, B) with $c \equiv \bar{c} \equiv 0$ has solutions with characteristic k if poles are allowed. (iv) If $n = -\nu$ and $k = -\kappa$ with $0 \leqq \kappa < \nu$, then the system (A, B) with $c \equiv \bar{c} \equiv 0$ always has continuous solutions with characteristic k if $f(s)$ satisfies the condition in (ii). (v) If $n = -\nu \leqq 0$, $k = -\kappa$ with $\kappa > \nu$, and $c \equiv \bar{c} \equiv 0$, there are always solutions with characteristic k if $\kappa \leqq 2\nu$ and f satisfies the condition in (ii); in this case $2\nu - \kappa$ zeros for the solution (in $\mathfrak{F} - \mathfrak{R}$) may be prescribed.

C. B. Morrey, Jr. (Berkeley, Calif.)

8216:

de Rham, Georges. Solution élémentaire d'opérateurs différentiels du second ordre. Ann. Inst. Fourier. Grenoble 8 (1958), 337–366.

L'auteur détermine toutes les solutions élémentaires de l'opérateur $\Box = \sum_1{}^p \partial^2/\partial x_i{}^2 - \sum_{p+1}^n \partial^2/\partial x_i{}^2$ (où $p > 0$ et $n - p > 0$) qui sont invariantes par le groupe de toutes les transformations linéaires de R^n en lui-même laissant invariants $u = \sum_1{}^p x_i{}^2 - \sum_{p+1}^n x_i{}^2$. Soit $f: x \in R^n \to u \in R^1$ et soit f^* l'opération de l'image transposé par f (qui fait correspondre à toute fonction $\psi(\xi)$ définie dans R^1 la fonction $f^*\psi(x) = \psi(u)$ définie dans R^n); f^* peut être définie pour chaque distribution dans R^1 qui s'annule au voisinage de 0. Soit $H_\varepsilon{}^k = f^*\delta_\varepsilon{}^{(k)}$, où $\delta_\varepsilon(\xi) = \delta(\xi - \varepsilon)$ et soit $Y(\xi)$ la fonction d'Heaviside; on pose

$$T_1 = \operatorname*{Pf}_{\varepsilon \to +0} Y(u-\varepsilon)u^{(2-n)/2}, \quad T_2 = \operatorname*{Pf}_{\varepsilon \to -0} Y(\varepsilon - u)|u|^{(2-n)/2}$$

lorsque n est impair $\geqq 3$ et

$$T_1 = \operatorname*{Pf}_{\varepsilon \to +0} Y(u-\varepsilon)u^{(2-n)/2} + \operatorname*{Pf}_{\varepsilon \to -0} Y(\varepsilon - u)u^{(2-n)/n},$$

$$T_2 = \lim_{\varepsilon \to 0} H_\varepsilon{}^{(n-4)/2}$$

lorsque n est pair $\geqq 4$. Suivant que $p(n-p)$ est pair ou impair, $E = C_1 T_1$ ou $E = C_2 T_2$ (il y a des formules pour les constantes C_1 et C_2) est une solution élémentaire (invariante): $\Box E = \delta_0$, et $E + aT_2 + b$ ou $E + aT_1 + b$ (a, b constantes) est la solution élémentaire (invariante) générale. En outre, l'auteur montre que T_1 et T_2 sont égales, à des facteurs constantes (il y'en a de formules) près, à $\Box^{(n-3)/2} Y(u)u^{-1/2}$ et $\Box^{(n-3)/2} Y(-u)|u|^{-1/2}$ ou à $\Box^{(n-2)/2} \log |u|$ et $\Box^{(n-2)/2} Y(u)$, suivant que n est impair ou pair. La méthode (appliquée par P.-D. Méthée [Comment. Math. Helv. **28** (1954), 225–269; MR **16**, 225] dans le cas où $p = 1$) consiste à utiliser les développements asymptotiques des $H_\varepsilon{}^k$ (pour $\varepsilon \to +0$ et $\varepsilon \to -0$ suivant le système des fonctions ε^h, $\varepsilon^h \log |\varepsilon|$) qui permettent d'obtenir les formules et de calculer les parties finies en question; la plupart de l'article est consacrée à ces développements asymptotiques.

S. Łojasiewicz (Kraków)

8217:

Greco, Donato. Le matrici fondamentali di alcuni sistemi di equazioni lineari a derivate parziali di tipo ellittico. Ricerche Mat. 8 (1959), 197–221.

This paper is concerned with the existence and properties of fundamental solution matrices and of solutions for second order elliptic systems of the form $(x = x_1, \cdots, x_n)$

$$\text{(1)} \qquad \sum_{r,s} a_{rs}{}^{(i)}(x) \frac{\partial^2 u_i}{\partial x_r \partial x_s} + \sum_j \left[\sum_s b_{ijs}(x) \frac{\partial u_j}{\partial x_s} + C_{ij}(x) u_j \right] = f_i(x).$$

A number of existence theorems "in the small" are proved assuming merely that the coefficients $\in C^{(0,\lambda)}$ for some λ with $0 < \lambda < 1$. The following uniqueness theorem is proved: If the a coefficients $\in C^1$ and the a, e, and c coefficients are uniformly in $C^{0,\lambda}$ for all x, where $e_{ijs} = b_{ijs} - \delta_{ij} \sum_r \partial a_{rs}{}^{(i)}/\partial x_r$, and if the quadratic form

$$\text{(2)} \qquad -\sum_i \sum_{r,s} a_{rs}{}^{(i)}(x) \lambda_r{}^{(i)} \lambda_s{}^{(i)} + \sum_{i,j} \sum_s e_{ijs}(x) \mu_i \lambda_s{}^{(j)} + \sum_{i,j} c_{ij}(x) \mu_i \mu_j$$

is negative definite for every x, then there exists at most one solution of (1) defined everywhere for which $u_i \partial u_i/\partial x_r = O(|x|^{1-n-\alpha})$ for some $\alpha > 0$. A number of other detailed theorems are proved concerning the existence and/or uniqueness of fundamental matrices and/or solutions of (1).

C. B. Morrey, Jr. (Berkeley, Calif.)

8218:

Greco, Donato. Su un problema al contorno per certi sistemi di equazioni ellittiche. Ricerche Mat. 8 (1959), 271–299.

This paper considers rather general boundary value problems for elliptic systems of equations of the type considered by the same author in the preceding review. In the case of n dimensions, uniqueness theorems are proved for the Dirichlet problem and a generalized Neumann problem in which

$$\mathscr{L}^{(j)}[u] \equiv a(x) \frac{du_j}{d\nu_j} + \sum_h \beta_{jh}(x) u_h = \varphi_j(x)$$

on the boundary, ν_j being the conormal corresponding to the jth operator. It is assumed that T is of class $A^1(C^1)$, φ_j is of class C', the leading coefficients $a_{rs}{}^{(j)} \in C^1$, the quadratic form (2) in the review mentioned above is negative definite, and the quadratic form $\beta_{jh}\mu_j\mu_h$ is positive definite in the case of the Neumann problems. Certain generalized volume, single-layer, and double-layer potentials are introduced and their differentiability properties are studied. In the case $n = 2$, more general boundary value problems involving the oblique derivative are studied by the method of singular integral equations. It is shown that the homogeneous integral equations have only a finite number of linearly independent solutions, and a necessary and sufficient condition for the existence of the nonhomogeneous equations is given in terms of the solutions of the adjoint homogeneous system of integral equations.

C. M. Morrey, Jr. (Berkeley, Calif.)

8219:

Avantaggiati, Antonio. Su un problema al contorno per un sistema ellittico di equazioni lineari alle derivate parziali del prim'ordine in tre variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 332–335.

Given a vector function $u=(u_0, u_1, u_2, u_3)$, the four 3-vectors $\mathbf{U}_0$, $\mathbf{U}_k$ are defined by $\mathbf{U}_0=u_1\mathbf{i}_1+u_2\mathbf{i}_2+u_3\mathbf{i}_3$, $\mathbf{U}_k=\mathbf{i}_k\times\mathbf{U}_0-u_0\mathbf{i}_k$ $(k=1, 2, 3)$, where $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ constitute a normal orthogonal set. The quadratic forms $F^k(u)=(u_k\mathbf{U}_0-u_0\mathbf{U}_k+\mathbf{U}_k\times\mathbf{U}_0)\cdot n$ are defined on the boundary ∂T of the domain T, assumed to be of class $C^{1+\lambda}$, $0<\lambda<1$; $F^{(k)}(u, v)$ denotes the corresponding bilinear form. The paper states the following theorem and indicates the principal steps in its proof. Suppose the 4-vector f to be of class $C^{0+\lambda}$ on $T\cup\partial T$, and the 4-vectors a and b and the functions α and β to be of class $C^{0+\lambda}$ on ∂T. Then there exists a unique 4-vector z of class $C^{0+\lambda}$ on $T\cup\partial T$ and of class C^1 on T such that grad z_0 + rot $\mathbf{Z}_0=\mathbf{F}_0$ and div $\mathbf{Z}_0=f_0$ in T, $a_0z_0+\mathbf{A}_0\cdot\mathbf{Z}_0=\alpha$ and $b_0z_0+\mathbf{B}_0\cdot\mathbf{Z}_0=\beta$ on ∂T, provided that a and b satisfy $F^{(k)}(a)\cdot F^{(k)}(b)-[F^k(a, b)]^2>0$ on ∂T for at least one index k. *C. M. Morrey, Jr.* (Berkeley, Calif.)

8220:

Zitarosa, Antonio. Su alcuni sistemi iperbolici di equazioni a derivate parziali del primo ordine. Ricerche Mat. **8** (1959), 240–270.

The author investigates the existence of solutions of the hyperbolic system $\partial u/\partial x=f(x, y, u, v)$, $\partial v/\partial y=g(x, y, u, v)$, $u(0, y)=v(x, 0)=0$, $0\leqq x, y\leqq 1$ and of similar systems by means of the fixed-point theorem for compact operators in the Banach space $C([0, 1]\times[0, 1])$. This method was previously used by C. Ciliberto [Ricerche Mat. **4** (1955), 15–29; MR **17**, 621]. The compactness of the operator involved hinges on a "uniform" uniqueness and continuous dependence criterion for the ordinary differential equation

$$z' = f(t, z, \lambda, \varphi(t, \lambda)), \quad z(0, \lambda) = 0,$$

where λ is a parameter and $\varphi(t, \lambda)$ is continuous in the unit square. Various sufficient conditions are given for this uniqueness and continuous dependence; the more concrete ones are "uniform" forms of generalized Osgood-Perron criteria and of the separable-variables case for ordinary differential equations. *J. J. Schäffer* (Montevideo)

8221:

Suschowk, Dietrich. Verallgemeinerte Ausstrahlungslösungen der Wellengleichung bei beliebiger Dimensionszahl. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 387–413 (1960).

The object of this paper is the construction of a solution of the wave equation $\sum_{i=1}^n \partial^2U/\partial x_i^2=\partial^2U/\partial t^2$ in Euclidean space of n dimensions of the form $u(t, r)Y(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ in spherical polar coordinates $(r, \alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ in the region $t\geqq r\geqq 0$ and vanishing on the characteristic half-cone $t=r$, $t\geqq 0$. Functions u of the form $\int_1^{t/r} A(\tau)f(t-\tau r)d\tau$ are sought, mainly (but not always) in the case when the separation constant has one of the admissible values $l(l+n-2)$ where $l=0, 1, 2, \cdots$. The cases n even and n odd have to be treated separately. *E. T. Copson* (St. Andrews)

8222:

Lax, Peter D. The scope of the energy method. Bull. Amer. Math. Soc. **66** (1960), 32–35.

The author applies the energy method to discuss the stability of a class of explicit difference approximations to linear hyperbolic equations with variable coefficients. Consider first the scalar case. Let (1) $u_t=Au_x$ be approximated by (2) $v_k=\sum_{j=-N}^{N} c_j(k)u_{k-j}$, where u_j and v_j denote the solution of the difference equation at position $j\Delta x$ and times t and $t+\Delta$, respectively. For consistency assume that (3) $\sum c_j(k)=1$. Let (4) $C_k(\theta)=\sum c_j(k)e^{ij\theta}$, and assume that (5) $|C_k(\theta)|\leqq 1$ at each point. Then, he outlines a proof of the following result. If the coefficients are Lipschitz continuous, if the multiplicity of the roots of $1-|C_k(\theta)|^2$ changes at most by two as k varies, and if (5) holds, then the difference equation is stable. The author summarizes generalizations to the case that (1) is a system. *J. Douglas, Jr.* (Houston, Tex.)

8223:

Èĭdel'man, S. D. On regular and parabolic systems of partial differential equations. Amer. Math. Soc. Transl. (2) **16** (1960), 464–468.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 254–258 [MR **19**, 150].

8224:

Godunov, S. K. An instance of nonuniqueness for a nonlinear parabolic system. Dokl. Akad. Nauk SSSR **136** (1961), 1281–1282 (Russian); translated as Soviet Math. Dokl. **2**, 188–189.

An example is constructed in which two different pairs of bounded functions (u, v) each satisfy the system $\partial u/\partial t=\partial[a(u, v)\partial u/\partial x]/\partial x$, $\partial v/\partial t=\partial^2v/\partial x^2$ and the initial data $u(x, 0)=v(x, 0)=1$ if $x>0$ and $=-1$ if $x<0$. *R. N. Goss* (San Diego, Calif.)

8225:

Ventcel', T. D. A free boundary problem for the heat equation. Dokl. Akad. Nauk SSSR **131** (1960), 1000–1003 (Russian); translated as Soviet Math. Dokl. **1**, 358–361.

Conditions are found for the existence and uniqueness of the solution, $u(x, t)$ and $s(t)$, to the equation

$$u_{xx} = u_t$$

in the region D, $0<x<s(t)$, $0<t<T$, $s(0)=0$, satisfying the boundary conditions

$$u(0, t) = f_1(t), \quad u[s(t), t] = f_2(t), \quad u_x[s(t), t] = g(t).$$

These conditions are given in the two following theorems.

Existence theorem: Let the functions f_1, f_2 and g be in C^2 and satisfy the conditions

$$\begin{aligned} &f_1 \leqq 0, \quad f_2 \leqq 0, \quad g > 0, \quad f_1(0) = 0, \\ (1)\qquad &f_2-f_1 > 0 \quad \text{for } t > 0, \quad f_1' \leqq 0, \quad f_2' \leqq 0; \\ &f_2'' \geqq 0, \quad (f_2-f_1)' \geqq 0, \quad g' \leqq 0. \end{aligned}$$

Then there exists a solution $u(x, t)$ and $s(t)$ such that $u(x, t)$ and its derivatives are continuous in the closure of D. The function $s(t)$ is differentiable, $s(0)=0$, $s(t)>0$ for $t>0$, and $s'(t)\geqq 0$.

Uniqueness theorem: The solutions $u(x, t)$ and $s(t)$ are unique if the following conditions hold: the function

$u(x, t)$ and its derivatives are continuous in the closure of D; $s(t)$ is differentiable and satisfies $s(t)>0$ for $t>0$, $s'(t)\geq 0$; and the functions f_1, f_2, f_1', f_2' and g are continuous and satisfy (1).

The existence is proven by considering the solution to the system of difference-differential equations

$$\frac{d^2}{dx^2}u(x, n\Delta t),\ \frac{u(x, n\Delta t)-u[x, (n-1)\Delta t]}{\Delta t}$$

for the boundary conditions

$$u(0, n\Delta t) = f_1(n\Delta t),\quad u[s(n\Delta t), n\Delta t] = f_2(n\Delta t),$$

$$\frac{d}{dx}u[s(n\Delta t), n\Delta t] = g(n\Delta t).$$

Uniqueness is obtained in the usual manner by assuming two solutions, u_1, s_1, and u_2, s_2 and showing that $u_1-u_2\equiv 0$ and $s_1-s_2\equiv 0$ in the closure of D.

G. W. Evans, II (Menlo Park, Calif.)

8226:

Mihaĭlov, V. Potentials of parabolic equations. Dokl. Akad. Nauk SSSR **129** (1959), 1226–1229. (Russian)

Consider the boundary value problem (1) $\partial u/\partial t=(-1)^{p-1}\partial^{2p}u/\partial x^{2p}$ for $0<x<1$, $0<t<T$; (2) $u(x, 0)=0$ for $0\leq x\leq 1$; (3) $\partial^k u(0, t)/\partial x^k=F_k(t)$, $\partial^k u(1, t)/\partial x^k=G_k(t)$ for $0\leq k\leq p-1$, $0\leq t\leq T$. The author sketches the proof of the existence of a solution of (1)–(3), provided some differentiability assumptions are made on F_k, G_k. The method consists of (i) introducing Green's function in a closed form, (ii) writing the solution as a sum of potentials with, as yet, unspecified densities, (iii) deriving integral equations for the densities. The interesting point is that the equations obtained are of the classical Volterra type. In (iii), fractional derivatives are used. {It should be remarked that results and methods of the present paper have already been obtained (independently and one year earlier) by R. K. Juberg and appeared in full detail (for $p=2$) in Pacific J. Math. **10** (1960), 859–878.

A. Friedman (Minneapolis, Minn.)

8227:

Trèves, François. Differential polynomials and decay at infinity. Bull. Amer. Math. Soc. **66** (1960), 184–186.

Un polynome $P(z)=P(z_1, \cdots, z_n)$, $z_i\in l$, est de type I [resp. II] si tous [resp. au moins un de] ses facteurs irréductibles admet[tent] des zéros réels. On désigne, avec L. Schwartz [*Théorie des distributions*, t. II, Hermann, Paris, 1951; MR **12**, 833], par $\mathcal{O}_c'$ l'espace des distributions sur R^n opérant sur $\mathcal{S}'$ (distributions tempérées) par convolution. L'auteur montre que P de type I $\Leftrightarrow$ pour toute distribution $T\in\mathcal{O}_c'$ telle que $P(\partial/\partial x_1, \cdots, \partial/\partial x_n)T=P(D)T$ soit à support compact, T est à support compact; P de type II $\Leftrightarrow$ il existe $F\in\mathcal{O}_c'$ telle que $P(D)F$ ait pour support l'origine. Exemples d'opérateurs de type I.

J. L. Lions (Nancy)

8228:

Žegalov, V. I. Boundary value problem for a mixed-type equation of higher order. Dokl. Akad. Nauk SSSR **136** (1961), 274–276 (Russian); translated as Soviet Math. Dokl. **2**, 45–47.

The problem is to solve the equation

$$\delta^{(n)}u \equiv (\partial^2/\partial x^2+\operatorname{sgn} y\,\partial^2/\partial y^2)^n u = 0$$

in a domain of the complex plane bounded by (i) a Jordan curve σ lying in the upper half-plane and terminating at $A(0, 0)$ and $B(1, 0)$ and (ii) the two characteristics AC: $x+y=0$ and CB: $x-y=1$. The solution $u(x, y)$ is to possess certain prescribed continuity and differentiability properties and to satisfy the following boundary conditions on σ and on one of the characteristics (say AC): $\delta^{(k)}u=\phi_k(\tau)$ on σ, $\delta^{(k)}u=\psi_k(x)$ on AC, $\phi_k(0)=\psi_k(0)$, $k=0, 1, \cdots, n-1$. The problem is reduced to a succession of Tricomi problems, solutions of which are obtained with the aid of the theory of polyanalytic functions.

R. N. Goss (San Diego, Calif.)

8229:

Kalašnikov, A. S. Uniqueness of the solution of the Cauchy problem for a class of quasi-linear hyperbolic systems. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 195–202. (Russian)

The uniqueness of generalized solutions of the initial value problem of

$$u_{it}+\phi_{ix}(t, x, u)+\psi_i(t, x, u) = 0 \quad (i = 1, 2)$$

is proved when the vector function ϕ of u is such that a certain mean value theorem holds. Assuming two solutions with the same initial values exist, the author derives an integral relation satisfied by any test function f vanishing for $t=T$, which is essentially an area integral only. A contradiction is obtained when f is the solution of an appropriate differential equation problem.

P. Ungar (New York)

8230:

Browder, Felix E. Eigenfunction expansions for non-symmetric partial differential operators. III. Amer. J. Math. **81** (1959), 715–734.

The present paper contains two distinct parts. The first is dedicated to the proof of the following important result: Let L be an elliptic differential operator with infinitely differentiable coefficients in the interior of an open set G of the n-dimensional real Euclidean space; suppose that each solution, in a connected open subset $G_0\subset G$, of $(L'-\zeta)u=0$ (where L' is the adjoint differential operator of L and ζ is an arbitrary complex number), which vanishes on an open subset of G_0, is identically zero throughout G_0. In this case, the eigenfunctions of L on G (i.e., the functions $u\in C^\infty(G)$ such that $(L-\zeta)^s u\equiv 0$ for some ζ and integer $s>0$) are dense in $C^\infty(G)$. The proof essentially uses a former result of the author concerning the completeness in L^2 of the eigenfunctions of L under null Dirichlet boundary conditions [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 433–439; MR **14**, 984]. The second part continues, in fact, the author's former articles on this subject [Amer. J. Math. **80** (1958), 365–381; **81** (1959), 1–22; MR **20** #1064; **21** #5821]. It contains some integral representations of positive definite kernels by means of the eigenfunctions of some differential operators which are not necessarily elliptic.

C. Foiaş (Bucharest)

8231:

Kreĭn, S. G. Differential equations in Banach space and their application in hydrodynamics. Amer. Math. Soc. Transl. (2) **16** (1960), 423–426.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 208–211 [MR **19**, 36].

POTENTIAL THEORY

See also 8112, 8196, 8207, 8280.

8232:

Fuglede, Bent. On the theory of potentials in locally compact spaces. Acta Math. **103** (1960), 139–215.

The main emphasis in this paper is placed on the study of capacitary distributions in a locally compact Hausdorff space X, with respect to consistent kernels; this last notion will be defined later. A lower semi-continuous function $k=k(x, y)>-\infty$ on $X\times X$ is called a kernel, and the potential $k(x, \mu)$ of a non-negative (Radon) measure μ with kernel k is defined by $\int k(x, y)d\mu(y)$; unless $k\geqq 0$, μ is assumed to have a compact support $S(\mu)$. The mutual energy $k(\mu, \nu)$ of μ and ν means $\int k(x, \nu)d\mu(x)$ and the set of non-negative measures μ with finite energy $k(\mu, \mu)$ is denoted by $\mathscr{E}^+$. Chapter I is of a preparatory character, and the methods and most of the results are well-known. After a survey over the relevant parts of the theory of measures and integration on a locally compact space, there follows an exposition of the theory of capacity and of the capacitary distributions on compact sets.

In chapter II, k is supposed to be positive definite: $k(\mu-\nu, \mu-\nu)=k(\mu, \mu)+k(\nu, \nu)-2k(\mu, \nu)\geqq 0$ for every μ, $\nu\in\mathscr{E}^+$. The strong topology on $\mathscr{E}^+$ is introduced by the semi-norm $\|\mu-\nu\|=(k(\mu-\nu, \mu-\nu))^{1/2}$. The notion of consistent kernel plays an important role in this paper. A kernel is called consistent if a strong Cauchy filter on $\mathscr{E}^+$ converging vaguely to μ converges strongly to μ. Several sufficient conditions for a positive definite kernel to be consistent are given, and it is shown that if k is a consistent kernel and $k(\mu, \mu)>0$ for every $\mu\in\mathscr{E}^+$, $\mu\not\equiv 0$, then $\mathscr{E}^+$ is strongly complete, i.e., any strong Cauchy filter on $\mathscr{E}^+$ converges strongly in $\mathscr{E}^+$. Next let T be another locally compact Hausdorff space, $\tau\geqq 0$ be a fixed measure on T and $k(x, y, t)$ be a lower semi-continuous function on $X\times X\times T$. Then $k(x, y)=\int k(x, y, t)d\tau(t)$ is called a kernel obtained by a superposition, and it is shown that such a kernel is consistent if all $k(x, y, t)$ are so and X is a metrizable K_σ-set. In the latter half of this chapter interior and exterior capacitary distributions are discussed for consistent kernels. For $A\subset X$ we set $1/\mathrm{cap}_* A=\inf k(\mu, \mu)$ for unit measures μ with $S(\mu)\subset A$, and $\mathrm{cap}^* A=\inf \mathrm{cap}_* G$ for open sets $G\supset A$. A measure $\lambda\in\mathscr{E}^+$ is called an interior capacitary distribution associated with A if $\|\lambda\|^2=\lambda(X)=\mathrm{cap}_* A$, $k(x, \lambda)\geqq 1$ in A except on a set B with $\mathrm{cap}_* B=0$ and $k(x, \lambda)\leqq 1$ everywhere on $S(\lambda)$. The existence is proved for A with $\mathrm{cap}_* A<\infty$ in theorem 4.1, and theorem 4.2 asserts that, as A_n increases to A, $\mathrm{cap}_* A_n$ goes to $\mathrm{cap}_* A$ and any interior capacitary distribution associated with A_n converges strongly to that associated with A provided $\mathrm{cap}_* A<\infty$. Correspondingly, an exterior capacitary distribution is defined, and the facts corresponding to theorems 4.1 and 4.2 are proved under the additional condition that X is normal and every open set is of class F_σ. This is applied to conclude the capacitability of every K-analytic set in X. The chapter ends with various extensions; in particular, balayage is discussed.

Chapter III is devoted to the special case in which X is a group and the kernel is a convolution kernel, i.e., $k(x, y)=k(xy^{-1})$, where $k(x)$ is a given lower semi-continuous function on X. It is shown that any definite convolution kernel is consistent in case X is compact and that, if X is abelian and if $k(x)$ has the form $\check{h}*h$ where $h\geqq 0$ is lower semi-continuous on X, then the convolution kernel $k(xy^{-1})$ is consistent and $\mathscr{E}^+$ is complete. Several important kernels, including those of order α, are of this form. Some types of kernels studied by Kunugui, Ninomiya and Ugaheri are investigated further and the paper closes with a number of examples serving to illustrate various points in the theory.
M. Ohtsuka (Hiroshima)

8233:

Doob, J. L. A non-probabilistic proof of the relative Fatou theorem. Ann. Inst. Fourier. Grenoble **9** (1959), 293–300.

L'auteur démontre en pure théorie du potential un théorème important qu'il a établi à l'aide des probabilités [Bull. Soc. Math. France **85** (1957), 431–458; MR **22** #844].

On considère un espace de Green R pourvu de sa frontière de Martin R' et des notions de limite (lim sup, lim inf) en tout point de $R\cup R'$, soit avec la topologie $\mathscr{C}$ de Martin, soit avec une topologie dite fine, ce qui revient alors à des définitions utilisant l'effilement de L. Naïm.

L'auteur, perfectionnant les résultats de L. Naïm [mêmes Ann. **7** (1957), 183–281; MR **20** #6608], montre que si une fonction g a une limite sup fine λ en un point frontière $\eta\in R'$ le long d'un ensemble $A\subset R$ (non effilé en η), g a une limite (selon $\mathscr{C}$) égale à λ le long d'un ensemble A_0 convenable $\subset A$ non effilé en η. Cela lui permet de compléter la théorie du problème de Dirichlet à la frontière de Martin et de montrer que (h étant harmonique >0 dans R), si f sur R' est h-résolutive, la "solution" (quotient par h d'une fonction harmonique) a une limite fine égale à f, μ^h-p.p. sur R' (μ^h étant la mesure correspondant à h dans la représentation de Martin). L'auteur en déduit que pour u quelconque surharmonique >0, u/h a une limite fine μ^h-p.p. sur R'. Puis il arrive au théorème fondamental qui est l'extension à h superharmonique >0 et à la limite μ^h-p.p. sur $R\cup R'$ (μ^h correspondant encore à h dans la représentation de Riesz-Martin).

Enfin une autre extension concerne l'allure à la frontière de u/h en affaiblissant la condition $u>0$.
M. Brelot (Urbana, Ill.)

8234:

Åkerberg, Bengt. Proof of Poisson's formula. Proc. Cambridge Philos. Soc. **57** (1961), 186.

The author rediscovers a well-known derivation of Poisson's integral formula for functions harmonic in a circle [cf., e.g., Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952; MR **13**, 640; p. 97].
R. P. Boas, Jr. (Evanston, Ill.)

8235:

Temko, K. V. Equilibrium potential, convex capacity and uniqueness of trigonometric series. Mat. Sb. (N.S.) **49 (91)** (1959), 109–132. (Russian)

Sei $\mathfrak{B}=\{B\}$ das System aller Borelschen Mengen eines gewissen Intervalls Γ des n-dimensionalen euklidischen Raumes ω. Das auf $\mathfrak{B}$ definierte Maß μ sei eine volladditive, nicht negative, normierte Mengenfunktion (d.h. $\mu(\Gamma)=1$). μ heißt auf $B\in\mathfrak{B}$ konzentriert, in Zeichen $\mu\prec B$, wenn $\mu(B)=1$. Sei r_{PQ} die euklidische Entfernung zweier Punkte P, $Q\in\omega$ und $\Phi(r_{PQ})$ eine positive Funktion

definiert für $0 \leq r < \infty$, stetig und abnehmend für $r > 0$. $u(P) = \int_\Gamma \Phi(r_{PQ}) d\mu(Q)$ definiert in jedem Punkt $P \in \omega$ ein Potential erzeugt durch μ. Das Existenzproblem für das Potential des Gleichgewichts heißt lösbar, für die beschränkte, abgeschlossene Menge $B \in \omega$, wenn ein Maß $\mu \prec B$ existiert derart, daß das durch dieses Maß erzeugte Potential konstant ist in jedem Punkt dieser Menge. Das durch dieses Maß erzeugte Potential heißt das Potential des Gleichgewichts. In dieser Arbeit wird eine Lösung des Problems des Gleichgewichts gegeben für eine lineare Menge B bestehend aus einer endlichen Zahl abgeschlossener Intervalle und für Funktionen $\Phi(r) = \int_0^{1/r} t|\lambda'(t)|dt$, wobei $\lambda(t) \in A$, d.h. $\lambda(t)$ ist eine positive, konvexe, zweimal differenzierbare Funktion mit $\lambda(t) \downarrow 0$ $(t \to \infty)$, $t|\lambda'(t)|$ nicht wachsend und $\int_0^\infty \lambda(t)dt = \infty$. Das Maß μ wird durchweg als stetig vorausgesetzt, d.h., $\mu(B) \to 0$ mit dem Durchmesser von B. $v(P) = \int_\Gamma \Phi(r_{PQ}) d\mu(Q)$ heißt das v-Potential erzeugt durch μ. Dabei ist Γ die Begrenzung der Einheitskugel. Satz 1: Bestehe die Menge E aus einer endlichen Zahl von getrennten Intervallen. Dann existiert ein Maß $\mu \prec E$ derart, daß das v-Potential erzeugt durch μ konstant ist auf E. — Im zweiten Teil der Arbeit wird die Frage der Kapazität von Mengen untersucht. Die Beweise der diesbezüglichen Sätze beruhen auf Methoden die zuerst von O. Frostman angewandt wurden [s. Dissertation, Lund, 1935]. In einer früheren Arbeit des Verfassers [Dokl. Akad. Nauk SSSR **110** (1956), 943–944; MR **19**, 31] wurde der Begriff der konvexen Kapazität folgendermaßen erklärt: Sei $Q(x) = \sum_{n=0}^\infty \lambda_n \cos nx$ mit $\lambda(t) \in G$, wobei $\lambda(t) \in G$, wenn $\lambda(t)$ positiv, konvex, $\lambda(t) \downarrow 0$ $(t \to \infty)$, $\int_0^\infty \lambda(t)dt = \infty$, $\Delta^2\lambda(0) > 0$, wo $\Delta^2\lambda(t) = \Delta\lambda(t) - \Delta\lambda(t+1)$, $\Delta\lambda(t) = \lambda(t) - \lambda(t+1)$ und $\lambda(n) = \lambda_n$. Sei ferner $u(x, r) = \int_B Q(x-y, r) d\mu(y)$ wo $B \in \mathfrak{B}$, $\mu \prec B$ und $Q(x, r) = \lambda_0/2 + \sum_{n=1}^\infty r^n \lambda_n \cos nx$ $(0 \leq r < 1)$. $u(x, r)$ heißt das u-Potential erzeugt durch das Maß μ. Ferner sei in diesem Fall stets $\Gamma = [0, 2\pi]$. Die konvexe Kapazität der Menge $B \in \mathfrak{B}$ erzeugt durch $\lambda(t) \in G$ wird definiert durch $C(B, \lambda) = e^{-\bar{V}_B}$, wobei $\bar{V}_B = \inf_{\mu \prec B}\{\lim \sup_{r \to 1-0} \sup_{0 \leq x \leq 2\pi} u(x, r)\}$. Demgegenüber wird hier nun die konvexe Kapazität der Borelmenge B erzeugt durch die Funktion $\lambda(t) \in G$ definiert durch $C(B, \lambda) = e^{-V_B}$, wobei

$$V_B = \inf_{\mu \prec B}\{\sup_{0 \leq x \leq 2\pi} u(x)\},$$

wobei $u(x) = \int_B Q(x-y) d\mu(y)$, $\mu \prec B$. (Wegen der Äquivalenz des erstgenannten Begriffs der konvexen Kapazität mit den klassischen Begriffen der α-Kapazität und der logarithmischen Kapazität sei auf die kürzlich erschienene Arbeit des Verfassers [Mat. Sb. (N.S.) **43** **(85)** (1957), 401–408; MR **20** #3414] verwiesen.) In Satz 2 wird nun bewiesen, daß für beliebige Borelmengen $V_B = \bar{V}_B$. Ferner wird gezeigt, daß der durch $C(B, \lambda) = e^{-\bar{V}_B}$ erklärte Kapazitätsbegriff äquivalent ist mit dem von Frostman folgendermaßen eingeführten Kapazitätsbegriff: Sei $I(\mu) = \iint_\Gamma u(P) d\mu(P)$ und $W_B = \inf_{\mu \prec B} I(\mu)$ dann ist die Kapazität $\mathfrak{C}$ der Menge B definiert durch $W_B = \Phi(\mathfrak{C})$ (für $\lambda(t) = t^{\alpha-1}$ wurde dies von Frostman gezeigt.) Ferner wird ein neuer Begriff der konvexen Kapazität folgendermaßen erklärt: Sei $I(\mu) = \int_B u(x) d\mu(x)$, wobei $\mu \prec B$, $u(x) = \int_B Q(x-y) d\mu(y)$ und sei $W_B = \inf_{\mu \prec B} I(\mu)$. Dann ist die konvexe Kapazität der Borelmenge B erklärt als die Zahl $C(B, \lambda) = e^{-W_B}$. Verschiedene Sätze über die Positivität dieser konvexen Kapazität werden bewiesen. Z.B. (Satz 3): Für die Positivität der konvexen Kapazität der Menge $B \in \mathfrak{B}$ im Sinn der letzten Definition ist notwendig und hinreichend, daß ein Maß $\mu \prec B$ existiert derart, daß $\sum_{n=1}^\infty (\alpha_n^2 + \beta_n^2)\lambda_n < \infty$, wo α_n und β_n die Fourier-Stieltjes-Koeffizienten von $d\mu$ sind. — Im letzten Teil der Arbeit wird der Begriff der konvexen Kapazität zur Bestimmung der Eindeutigkeitsmengen trigonometrischer Reihen benützt. $(a_n, b_n) \equiv a_0/2 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) \in T_\alpha$, wenn für $0 < \alpha < 1$, $\sum_{n=1}^\infty (a_n^2 + b_n^2) n^{-\alpha} < \infty$. $B \in U\{T_\alpha\}$, d.h., B gehört zur Klasse der Eindeutigkeitsmengen bezüglich der Klasse der Reihen $\in T_\alpha$, wenn aus $(a_n, b_n) \in T_\alpha$ und $f(x, r) = a_0/2 + \sum_{n=1}^\infty r^n(a_n \cos nx + b_n \sin nx) \to 0$ $(r \to 1-0)$ außerhalb B folgt: $a_n = b_n = 0$ $(n = 0, 1, 2, \cdots)$. Broman [Dissertation, Univ. Uppsala, Uppsala, 1947; MR **9**, 182] bewies: Dann und nur dann ist die abgeschlossene Menge $B \in U\{T_\alpha\}$, wenn ihre α-Kapazität Null ist. In der vorliegenden Arbeit wird ein entsprechender Satz für eine wesentlich größere Klasse von Reihen bewiesen. $(a_n, b_n) \in T_\lambda$ mit $\lambda(t) \in A$, wenn $\sum_{n=1}^\infty (a_n^2 + b_n^2)\lambda_n < \infty$. Entsprechend wie $U\{T_\alpha\}$ wird $U\{T_\lambda\}$ mit $\sum_{n=1}^\infty (a_n^2 + b_n^2)\lambda_n$ anstelle von $\sum_{n=1}^\infty (a_n^2 + b_n^2) n^{-\alpha}$ definiert. Ferner sei $\lambda(t) \in D$, wenn $\lambda(t) \in A$ und $t\,\lambda(t) \uparrow \infty$ $(t \to \infty)$, $t^2|\lambda'(t)|$ nicht abnehmend für $t \to \infty$ und $3|\lambda(t)| - t\lambda''(t)$ nicht negativ und nicht abnehmend für $t \to \infty$. Satz 8. Hinreichend dafür, daß die abgeschlossene Menge $B \subset U\{T_\lambda\}$, ist, daß die konvexe Kapazität der Menge B erzeugt durch die Funktion $\lambda(t) \in G$ Null ist; wenn $\lambda(t) \in D$, dann ist diese Bedingung auch notwendig. Für $\lambda(t) = t^{\alpha-1}$ $(0 < \alpha < 1)$ ist dies die Aussage von Broman.

G. Goes (Evanston, Ill.)

8236:

Matschinski, Matthias. Les moyennes invariantes des solutions de quelques équations aux dérivées partielles. C. R. Acad. Sci. Paris **250** (1960), 2504–2506.

Let $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ be the mean values of the function φ over a spherical surface and a solid sphere of radius R about a variable point in m-space. If φ is real analytic, then

$$(*) \quad \bar{\varphi} = \sum_{n=1}^{\infty} C_{m,2n} R^{2n} \Delta^n \varphi, \quad \bar{\bar{\varphi}} = \sum_{n=1}^{\infty} \frac{m}{2n+m} C_{m,2n} R^{2n} \Delta^n \varphi,$$

where Δ is the Laplacian and where

$$C_{m,2n} = \frac{1}{2 \cdot 4 \cdots (2n)} \cdot \frac{1}{m(m+2) \cdots (m+2n-2)}.$$

The author applies $(*)$ to obtain mean value theorems for polyharmonic functions and for solutions of $\Delta\varphi \pm a^2\varphi = 0$.

W. Noll (Pittsburgh, Pa.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 8341.

8237:

Anastassiadis, Jean. Remarques sur quel[illegible]es équations fonctionnelles. C. R. Acad. Sci. Paris **250** (1960), 2663–2665.

Ergebnisse von E. Artin [*Einführung in die Theorie der Gammafunktion*, Teubner, Leipzig, 1931], A. E. Mayer [Acta Math. **70** (1938), 57–62] und H. P. Thielman [Bull. Amer. Math. Soc. **47** (1941), 118–120; MR **2**, 311] verallgemeinernd, beweist Verf., dass $f(x) = \Gamma(x/\omega)p\omega^{-p+px/\omega}$ die einzige logarithmisch konvexe Lösung der Differenzengleichung $(*)\, f(x+\omega) = x^p f(x)$, $(p > 0, \omega > 0)$ mit $f(\omega) = 1$, $f(x) > 0$ für $x > 0$, während $f(x) = (2\omega)^{-p/2}\Gamma(x/2\omega)^p\Gamma((x+\omega)/2\omega)^{-p}$ die

einzige ω-abnehmende oder ω-konvexe Lösung der Gleichung $f(x+\omega)=x^{-p}/f(x)$ ist $(x>0,\ p>0,\ \omega>0)$, und weist darauf hin, dass auch

$$f(x) = c^{-1/2}(2\omega)^{-pk/2}\Gamma((x+\rho_1)/2\omega)^p\Gamma((x+\rho_1+\omega)/2\omega)^{-p}\cdots \Gamma((x+\rho_k)/2\omega)^p\Gamma((x+\rho_k+\omega)/2\omega)^{-p}$$

die einzige ω-abnehmende oder ω-konvexe Lösung der Differenzengleichung

$$f(x+\omega) = (1/c)(x+\rho_1)^{-p}\cdots(x+\rho_k)^{-p}/f(x)$$
$$(x>0,\ p>0,\ \omega>0,\ c>0,\ \rho_1>0,\ \cdots,\ \rho_k>0)$$

ist. Verf. nennt eine Funktion $f(x)$ ω-abnehmend, falls $f(x+\omega)\leqq f(x)$, und ω-konvex (von oben) falls $f(x)\geqq (f(x-\omega)+f(x+\omega))/2$ ist.

Bezüglich logarithmisch konvexen Lösungen von (*) und von ähnlichen Gleichungen vgl. auch W. Krull [Math. Nachr. **1** (1948), 365–376; **2** (1949), 251–262; MR **11**, 112, 364] und A. Dinghas [Math.-Phys. Semesterber. **6** (1959), 245–252].

J. Aczél (Debrecen)

8238:

Lesky, Peter. Über die Lösung linearer homogener Differenzengleichungen zweiter Ordnung mit Hilfe hypergeometrischer Funktionen. Monatsh. Math. **64** (1960), 272–288.

A difference equation

$$p_ny_{n+2}+q_ny_{n+1}+r_ny_n = 0 \quad (n=0,1,2,\cdots)$$

can be transformed into a standard form $\Delta^2v_n+I_nv_n=0$ $(n=0,1,2,\cdots)$ by a substitution $y_n=\varphi_nv_n$, where φ_n satisfies the first-order equation $2p_n\varphi_{n+2}+q_n\varphi_{n+1}=0$. We have $I_n=(4r_np_{n-1}-q_nq_{n-1})/q_nq_{n-1}$, and I_n is called the invariant. The author applies this to the case where $p_n=\alpha_1n+\alpha_2$, $q_n=\beta_1n+\beta_2$, $r_n=\gamma_1n+\gamma_2$. He shows that it has the same invariant as a well-known difference equation, viz.. Gauss' relation between $F_{n+\rho}$, $F_{n+1+\rho}$, $F_{n+2+\rho}$, where $F_a=F(a,b;c;z)$, provided that ρ, b, c and z are suitably chosen. In this way the solutions of the given equation can be expressed in terms of hypergeometric functions. Several examples are given.

N. G. de Bruijn (Eindhoven)

8239:

Kurepa, Svetozar. Functional equation $F(x+y)\times F(x-y)=F^2(x)-F^2(y)$ in n-dimensional vector space. Monatsh. Math. **64** (1960), 321–329.

In this paper F denotes a mapping of the real number field R into the family of all complex n-by-n matrices. The problem is to develop information about F if the determinant of $F(x)$ is nonzero for almost all x and if $F(x+y)F(x-y)=F^2(x)-F^2(y)$ for all x and y. Under a measurability hypothesis it is shown that the components of F are of class C^∞ and that F satisfies the differential equation $F'(x)F''(x)=F(x)F'''(x)$. The conditions $F(0)=0$, $F(-x)=-F(x)$, $F(x)F(y)=F(y)F(x)$ are also necessary. Using these conditions, the author determines the possibilities for F in the cases $n=1, 2, 3$. If F is further required to be uniformly bounded, the solution for these values of n is $F(x)=A\sin\alpha x$ (α real, A an arbitrary nonsingular constant matrix).

A. E. Taylor (Los Angeles, Calif.)

8240:

Ghermanescu, Michel. Sur la fonction $\sigma(u)$ de Weierstrass. C. R. Acad. Sci. Paris **251** (1960), 1710–1711.

The author outlines the proof of the following. The set of continuous and single-valued solutions of the functional equation

$$f(u+v)f(u-v)f(w+t)f(w-t)+f(u+w)f(u-w)f(t+v)\times f(t-v)+f(u+t)f(u-t)f(v+w)f(v-w) = 0$$

consists of the functions $C\exp(pu^2)\sigma(u)$ and $C\exp(pu^2)\times\sin(2\pi u/a)$, where C, p, a are arbitrary constants and $\sigma(u)$ is the Weierstrass elliptic function.

A. E. Danese (Schenectady, N.Y.)

SEQUENCES, SERIES, SUMMABILITY

See also 8274.

8241:

Thron, W. J. Sequences generated by iteration. Trans. Amer. Math. Soc. **96** (1960), 38–53.

Let g satisfy $0<g(x)<x$ in some interval $0<x<c$. If x_1 satisfies $0<x_1<c$, the sequence $x_1, x_2, x_3, \cdots$ is determined by $x_{n+1}=g(x_n)$ $(n=1, 2, 3, \cdots)$. Then $x_n\to0$, and the question as to the asymptotic behaviour arises. The author is exclusively interested in the behaviour as far as it is independent of the initial element. Apart from a discussion of known facts concerning the case where $g'(0)$ exists with $0\leqq g'(0)<1$, the author devotes his attention entirely to the case where $g'(0)=1$. Example: If k is a positive integer,

$$g(x) = x+\sum_{\nu=1}^{k+1}a_{k+\nu}x^{k+\nu}+R(x),$$
$$R(x) = O(x^{2k+1+\epsilon}),\quad a_{k+1}<0,$$

then he proves the existence of constants $c_1, c_2, \cdots, c_{k+1}$ with

$$x_n = \sum_{\nu=1}^{k}c_\nu n^{-\nu/k}+c_{k+1}n^{-(k+1)/k}\log n+O(n^{-(k+1)/k}),$$

and the influence of variations of x_1 is of the order $n^{-(k+1)/k}$.

The author remarks that the typical case $g(x)=\sin x$ was treated in detail in the reviewer's *Asymptotic methods in analysis* [North-Holland, Amsterdam, 1958; MR **20** #6003; chapter 8]. However, he seems to have been unaware of the fact that exercise 8.11 of that book contains a quite general case, from which most of the results in the paper can be derived.

N. G. de Bruijn (Eindhoven)

8242:

Maruki, Goichi. Some non-linear transformation. Bull. Fukuoka Gakugei Univ. III **9** (1959), 15–17.

On peut accélérer la convergence d'une suite

$$A_n = B+\sum_{i=1}^{R}a_iq_i^n \quad \text{ou} \quad A_n = B+\int_{q_1}^{q_2}a_0(q)q^n\,dq$$

en la remplaçant par l'une des suites

$$B_n = \begin{vmatrix} A_{n-1} & A_n \\ \Delta A_{n-1} & \Delta A_n \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 \\ \Delta A_{n-1} & \Delta A_n \end{vmatrix}.$$

J. Kuntzmann (Grenoble)

8243:

Åkerberg, Bengt. A variant on the proofs of some inequalities. Proc. Cambridge Philos. Soc. **57** (1961), 184–186.

The author gives shorter proofs of two inequalities related to Carleman's inequality [see MR **9**, 234 for a review of his earlier paper, Ark. Mat. Astr. Fys. **34B** (1947), no. 13].

R. P. Boas, Jr. (Evanston, Ill.)

8244:

Boas, R. P., Jr. Inequalities for monotonic series. J. Math. Anal. Appl. **1** (1960), 121–126.

The author establishes two theorems (of which the first is used in the proof of the second) concerning inequalities for sums (repeated, in theorem 2) of decreasing or nearly decreasing terms. Theorem 1 contains the basic result:

Suppose that $a_n \geqq 0$, $\lambda \geqq 0$, $n^{-\lambda}a_n$ is nonincreasing and $b > 1$. Then

$$\left(\sum_M^N a_k\right)^p \geqq A(p, \lambda, b) \sum_{Mb}^N a_k{}^p k^{p-1} \quad (p > 1)$$

or equivalently

$$\left(\sum_M^N a_k\right)^p \leqq A(p, \lambda, b) \sum_{M/b}^N a_k{}^p k^{p-1} \quad (0 < p < 1),$$

here, as throughout, A denoting a number dependent only on the indicated arguments (not always the same) and a sum with nonintegral limits extending over all the integers between the limits. Theorem 2, which generalises results obtained earlier by A. A. Konyuškov [Mat. Sb. (N.S.) **44** (86) (1958), 53–84; MR **20** #2571], reads:

Let $K(x, y)$ be homogeneous of degree -1, let $\int_0^\infty K(1, u)^p du < \infty$, and suppose that either (i) $K(m, n)$ decreases in both arguments; or (ii) $K(m, n) = 0$ for $m > n$ and $K(m, n)$ decreases in both arguments for $m \leqq n$; or (iii) $K(m, n) = 0$ for $m < n$ and $K(m, n)$ decreases in m and is monotonic in n for $m \geqq n$. Let $a_m m^{-\lambda}$ decrease for some non-negative λ. Then

$$\sum_{n=1}^\infty \left\{ \sum_{m=1}^\infty K(m, n) a_m \right\}^p \geqq A(p, \lambda, K) \sum_{m=1}^\infty a_m{}^p \quad (p > 1),$$

$$\sum_{n=1}^\infty \left\{ \sum_{m=1}^\infty K(m, n) a_m \right\}^p \leqq A(p, \lambda, K) \sum_{m=1}^\infty a_m{}^p \quad (0 < p < 1).$$

A number of applications of these results are indicated. In particular, the special case of the second inequality in theorem 2, with $K(m, n) = 1/n$ for $m \geqq n$, $K(m, n) = 0$ for $m < n$, $a_m{}^2$ instead of a_m and $p = 1/2$, is interpreted as equivalent to the following statement about the rapidity of convergence of a series of orthonormal functions $\phi_k(x)$ with monotonic coefficients: Let $f(x) \sim \sum a_k \phi_k(x)$ with partial sums $s_n(x)$, let $n^{-\lambda}a_n$ decrease and $\sum a_n$ converge. Then $\sum_{n=1}^\infty n^{-1/2} \|f(x) - s_n(x)\|$ converges, the norm being taken in L^2.

T. Pati (Allahabad)

8245:

Groemer, Helmut. Über den Minimalabstand der ersten *N* Glieder einer unendlichen Punktfolge. Monatsh. Math. **64** (1960), 330–334.

Let $\{P_m\}$ be a sequence of points on the unit interval $[0, 1]$, and write $d_N = \min |P_i - P_k|$ $(1 \leqq i, k \leqq N;\ i \neq k)$. The estimate

$$(*) \qquad \liminf_{N \to \infty} N d_N \leqq 1/\log 4$$

was established, in recent years, by a number of authors; cf. N. G. de Bruijn and P. Erdös [Nederl. Akad. Wetensch. Proc. **52** (1949), 14–17 = Indag. Math. **11**, 46–49; MR **11**, 423], A. Ostrowski [Arch. Math. **8** (1957), 1–10; MR **19**, 638], G. H. Toulmin [ibid. 158–161; MR **20** #37], A. Schönhage [ibid. 327–329; MR **20** #35]. In the present paper a generalization of the problem for n-dimensional euclidean space R_n is considered. Let $d(P, Q)$ be a normalized Minkowski metric for R_n, i.e., $d(P, Q)$ is a real-valued function defined on $R_n \times R_n$ and satisfying the following conditions. (i) $d(P, Q) > 0$ for $P \neq Q$. (ii) $d(P, Q) = d(R, S)$ for $P - Q = R - S$. (iii) $d(tP, tQ) = |t| d(P, Q)$ for all real t. (iv) $d(P, R) \leqq d(P, Q) + d(Q, R)$. (v) For some fixed point P_0, the set of points X such that $d(P_0, X) \leqq 1$ has volume 2^n. Further, let B be a bounded subset of R_n having Jordan volume 1. Finally, let $\{P_m\}$ be an arbitrary sequence of points in B, and write

$$d_N = \min d(P_i, P_k) \quad (1 \leqq i, k \leqq N;\ i \neq k).$$

It is then shown that

$$(**) \qquad \liminf_{N \to \infty} N d_N^n \leqq \left(1 + n \int_0^1 \frac{(1-x)^n}{x+1}\, dx\right)^{-1}.$$

For $n = 1$, the expression on the right-hand side of (**) is equal to 1/log 4. Thus the result proved in the present paper contains (*) as a special case. It is known that the constant in (*) is best possible; whether the same is true for (**) remains an open question.

{Reviewer's remark. The factor $(1-x)^n$ on the right-hand side of (**) is printed incorrectly as $(x-1)^n$.}

L. Mirsky (Sheffield)

8246:

Heller, Robert, Jr. Some convergence theorems for continued fractions. Proc. Amer. Math. Soc. **11** (1960), 805–811.

This paper is a continuation of the study of the convergence of the continued fraction

$$\text{(A)} \qquad \frac{1}{b_1} + \frac{a_1}{b_2} + \frac{a_2}{b_3} + \cdots,$$

where each of a and b is a complex number sequence. (1) A necessary and sufficient condition is given for absolute convergence of the even and odd parts of (A). This generalizes a result of W. T. Scott and H. S. Wall [Trans. Amer. Math. Soc. **47** (1940), 155–172; MR **1**, 217] and R. E. Lane and H. S. Wall [ibid. **67** (1949), 368–380; MR **11**, 244]. (2) The convergence of some particular continued fractions is proved with the use of (1). (3) A certain extension of the "parabola theorem" is considered.

E. Frank (Chicago, Ill.)

8247:

Dawson, David F. Concerning convergence of continued fractions. Proc. Amer. Math. Soc. **11** (1960), 640–647.

The continued fraction

$$F(c, d) = d_0 + \frac{c_0}{d_1} + \frac{c_1}{d_2} + \frac{c_2}{d_3} + \cdots$$

with sequence of approximants $\{F_p\}$ is said to possess property (A) if there exists a complex number v such that each of the sequences $\{F_{2p-1}\}$ and $\{F_{2p}\}$ contains an infinite subsequence convergent to v. Here c denotes the

complex number sequence $\{c_p\}_{p=0}^{\infty}$ and d denotes the complex number sequence $\{d_p\}_{p=0}^{\infty}$. Also $F(c, d)$ is said to be absolutely convergent at least in the wider sense if there exists a positive integer n such that the continued fraction

$$d_{n-1}+\frac{c_{n-1}}{d_n}\,\frac{c_n}{+d_{n+1}}\,\frac{c_{n+1}}{+d_{n+2}}+\cdots$$

converges absolutely. In this paper results of a study of these properties are presented and applied to the problem of convergence. Convergence criteria are found which improve some known results. For example, let

$$f(a)=\frac{1}{1+}\,\frac{a_1}{1+}\,\frac{a_2}{1+}\,\frac{a_3}{1+}\cdots.$$

If the even [odd] part of $f(a)$ converges absolutely at least in the wider sense, the odd [even] part converges, and $\lim|f_{2p}|=\infty$ [$\lim|f_{2p-1}|=\infty$], then the series $\sum|b_{2p}|$ [the series $\sum|b_{2p-1}|$] converges and $\limsup|b_1+b_3+\cdots+b_{2p-1}|<\infty$ [$\limsup|b_2+b_4+\cdots+b_{2p}|<\infty$], where $b_1=1$, $b_{p+1}=1/a_pb_p$, $p=1, 2, \cdots$, and f_p is the pth approximant of $f(a)$. This result is used to show that, if there exists a sequence $\{r_p\}_{p=1}^{\infty}$ of non-negative numbers such that $r_1|1+a_1|\geqq|a_1|$, $r_2|1+a_1+a_2|\geqq a_2$, $r_p|1+a_{p-1}+a_p|\geqq r_pr_{p-2}|a_{p-1}|+|a_p|$, $p=3, 4, \cdots$, with actual inequality holding either in the first relation or in the second, then either some $a_p=0$ and $f(a)$ converges, or else no $a_p=0$ and the divergence of the series $\sum|b_p|$ is necessary and sufficient for the convergence of $f(a)$.

E. Frank (Chicago, Ill.)

8248:

Srivastava, Pramila. On the summability of the Dirichlet's product of summable series. Arch. Math. **11** (1960), 342–345.

Verf. stellt den folgenden Satz über die absolute Cesàro-Summierbarkeit von Dirichletreihen auf. Für $\sigma>0$ sei $\sum a_nn^{-s}$ $|C, r_1|$-summierbar und $\sum b_nn^{-s}$ $|C, r_2|$-summierbar. Dann ist das Dirichletprodukt der beiden Reihen für $\sigma>\theta$ $|C, R|$-summierbar. Dabei sind $r_2\geqq r_1\geqq 0$, $0\leqq R\leqq r_1+r_2$; $\theta=r_2-R$ für $R\leqq r_2-r_1$, $=\frac{1}{2}(r_1+r_2-R)$ für $R\geqq r_2-r_1$. Weiter wird bewiesen, daß unter den angegebenen Bedingungen stets Dirichletreihen existieren, so daß die Produktreihe θ als Abszisse der $|C, R|$-Summierbarkeit hat, so daß der Satz in diesem Sinne bestmöglich ist. Die Beweise benutzen Resultate über die $|C, r|$-Abszissen und die Lindelöfsche μ-Funktion. *H.-E. Richert* (Göttingen)

8249:

Vernotte, Pierre. Sommation des séries divergentes à termes positifs, de valeur complexe. C. R. Acad. Sci. Paris **251** (1960), 1455–1456.

Suite d'une note antérieure [mêmes C. R. **250** (1960), 1785–1786; MR **22** #2813]. *J. Kuntzmann* (Grenoble)

8250:

Men'šov, D. E. On the summation of orthogonal series by linear methods. Dokl. Akad. Nauk SSSR **131** (1960), 507–509 (Russian); translated as Soviet Math. Dokl. **1**, 285–288.

Let $\sum a_n$ be a series with partial sums s_n; let $\{n_u\}$ be an increasing sequence of positive integers. The author says that $\sum a_n$ is summable $T[n_u]$ to s if $s_{n_u}\to s$. He calls two methods of summation equivalent with respect to the orthonormal system $\{\phi_n(x)\}$ if every L^2 Fourier series in the ϕ_n is either simultaneously summable by both methods to the same sum almost everywhere or simultaneously not summable almost everywhere. He constructs a totally regular matrix method B and a uniformly bounded orthonormal system $\{\phi_n(x)\}$ such that for every sequence $\{n_u\}$ the methods B and $T[n_u]$ are not equivalent with respect to $\{\phi_n(x)\}$. This contrasts with Kaczmarz's theorem [Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Monogr. Mat. Bd. 6, Warsaw-Lwów, 1935, p. 183] that if a Fourier series in the $\{\phi_n(x)\}$ is summable almost everywhere by a regular method B then it is summable almost everywhere by some $T[n_u]$. *R. P. Boas, Jr.* (Evanston, Ill.)

8251:

Geisberg, S. P. Certain properties of summability methods. Dokl. Akad. Nauk SSSR **137** (1961), 265–267 (Russian); translated as Soviet Math. Dokl. **2**, 256–259.

For summability methods A represented by matrices (a_{nk}), let A_1, A_2, A_3 denote the field of absolute null-summability, the field of boundedness, and the field of summability, respectively. Let c_0, m, e denote the sets of null-sequences, bounded sequences, and sequences $\{\xi_k\}$ with $\sum|\xi_k|<\infty$. For a subsequence $\phi=\{\phi_k\}$ of the sequence of natural numbers, the author denotes by A^ϕ the submatrix $(a_{n\phi_k})$ of A, and he announces fifteen theorems concerning special and general methods. Examples:

In order that $(C, \alpha)_1{}^\phi\subset e$ (where (C, α) denotes the Cesàro method of order α), it is necessary and sufficient that for some index s and some number $r>1$, $\phi_{k+1}>r\phi_k$ when $k>s$. For the Euler methods (E, α), the inequality on the ϕ_k becomes much weaker, namely, $\phi_{k+1}>\phi_k+r\sqrt{\phi_k}$; but an additional restriction on ϕ enters: for each $\{\xi_k\}$ in $(E, \alpha)_1{}^\phi$, there must exist a $\gamma>0$ such that $\xi_k=O(\gamma^k)$.

The following theorems are expressed in terms of norms. Let

$$\|\xi\|_{A_1}{}^p=\sum_n\Big|\sum_{k=p}^{\infty}a_{nk}\xi_k\Big|,$$

$$\|\xi\|_{A_2}{}^p=\|\xi\|_{A_3}{}^p=\sup_n\Big|\sum_{k=p}^{\infty}a_{nk}\xi_k\Big|,$$

and, for $i=1, 2, 3$, let $\|\xi\|_i{}^p=\|\xi\|_{I_i}{}^p$, where I denotes the identity matrix. If $A_1\subset m$, there exist an index p and a number $q>0$ such that $\|\xi\|_{A_1}{}^p>q\|\xi\|_2{}^p$ for all $\xi\in A_1\cap m$. If $A_i\subset m$ ($i=2$ or $i=3$), there exist an index p and a number $q>0$ such that

$$\|\xi\|_{A_i}{}^p>q\|\xi\|_2{}^p$$

for all $\xi\in A_i\cap m$.

Further theorems involve conjugates (of various orders) of linear operators on the Banach space c_0.

G. Piranian (Ann Arbor, Mich.)

8252:

Wirszup, Izaak. On an extension of Cesàro's methods of summability to logarithmic scales. J. Math. Mech. **9** (1960), 783–812.

Let $(1-x)^{-\alpha_0-1}\prod_{j=1}^{k}(\log_j(e_j/(1-x)))^{\alpha_j}=\sum A_n^{\alpha_0\alpha_1\cdots\alpha_k}$ where $e_1=e$, $e_j=e^{e_{j-1}}$ and α_j are real. Denote by $(C, \alpha_0, \alpha_1, \cdots, \alpha_k)$ the Nörlund method corresponding to the sequence $\{A_n^{\alpha_0\alpha_1\cdots\alpha_k}\}$. For $k=0$ the method is Cesàro of order α_0. The author deduces preliminary theorems for the

$(C, \alpha_0, \cdots, \alpha_k)$ methods from known theorems on Nörlund methods. An example of a further development of the subject is a Tauberian theorem: if $\sum u_n$ is $(C, 0, \cdots, 0, \beta)$-summable (k zeros, $\beta > 0$) and $u_n = o(n^{-1} \log n \log_2 n \cdots \log_k n)$, then $\sum u_n$ converges. Theorems on (C, α) summability of Fourier series are extended to the more general methods. *R. E. Williamson* (Cambridge, Mass.)

8253:

Newman, Donald J. $1-1+1-1+\cdots = 1/2$. Proc. Amer. Math. Soc. **11** (1960), 440–443.

Let $\{g(n)\}$ be a strictly increasing sequence of integers, $g(0) = 0$. The author compares the way in which various functions

$$f(t) = \sum_{n=0}^{\infty} (-1)^n t^{g(n)}$$

approach the limit $\frac{1}{2}$ as $t \uparrow 1$. He first takes $g(n) = n^k$ and shows that (i) for k odd, the difference $f(t) - \frac{1}{2}$ is as large as $c(1-t)$, $c > 0$; (ii) for k even, the difference is as small as $\exp\{-c/(1-t)^\alpha\}$, $\alpha = 1/(k-1)$, $c > 0$. He then points out that by a theorem of the reviewer, the exponents $g(n) = n^2$ give the "smallest possible" difference $f(t) - \frac{1}{2}$. More precisely, if

$$|f(t) - \tfrac{1}{2}| \leq \exp\{-\varphi(t)/(1-t)\},$$

with decreasing exponent, then $\varphi(t)$ must be bounded above as $t \uparrow 1$ [Korevaar, Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 36–45; MR **15**, 698]. *J. Korevaar* (Madison, Wis.)

8254:

Borwein, D. On strong and absolute summability. Proc. Glasgow Math. Assoc. **4**, 122–139 (1960).

Let $Q = (q_{n,r})$ be a summability matrix and let

$$\sigma_n = \sum_{r=0}^{\infty} q_{n,r} s_r,$$

where $s_n = \sum_{r=0}^{n} a_r$. For another matrix $P = (p_{n,r})$ with $p_{n,r} \geq 0$, if

$$P(|\sigma_n - s|^\lambda) = \sum_{r=0}^{\infty} p_{n,r} |\sigma_r - s|^\lambda$$

is defined for each n and tends to 0 as $n \to \infty$, then the author says that $\sum a_r$ is strongly summable $[P, Q]_\lambda$ to s. If

$$\sum_{n=1}^{\infty} n^{\gamma\lambda + \lambda - 1} |\sigma_n - \sigma_{n-1}|^\lambda < \infty$$

then the author says $\sum a_r$ is absolutely summable $|Q, \gamma|_\lambda$. These definitions are straightforward generalizations of known ones [J. M. Hyslop, same Proc. **1** (1952), 16–20; MR **14**, 368; T. M. Flett, Proc. London Math. Soc. (3) **8** (1958), 357–387; MR **21** #1481]. Some of the properties of the strong and absolute summability processes defined above are given. In particular the author investigates the case of the Hausdorff matrix concisely. *G. Sunouchi* (New Haven, Conn.)

8255:

Kangro, G. F. On linear and bilinear transformations of sequences in a Banach space. Amer. Math. Soc. Transl. (2) **16** (1960), 414–416.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 199–201 [MR **20** #4122].

8256:

Melencov, A. A.; Muraev, È. B. On the theory of summation of double series by Borel's methods. Dokl. Akad. Nauk SSSR **130** (1960), 1193–1195 (Russian); translated as Soviet Math. Dokl. **1**, 150–152.

Let $\sum_{i,k} a_{ik}$ be a double series with sequence of partial sums $\{A_{mn}\}$, $A_{mn} = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik}$. Let

$$A(x, y) = \sum_{i,k=0}^{\infty} A_{ik} \frac{x^i}{i!} \frac{y^k}{k!}, \quad a(x, y) = \sum_{i,k=0}^{\infty} a_{ik} \frac{x^i}{i!} \frac{y^k}{k!}$$

converge for all $x, y \geq 0$. The double series is said to be B_λ-summable to S if $e^{-(x+y)} A(x, y) \to S$ as $(x, y) \to \infty$ inside the sector $\lambda \leq y/x \leq 1/\lambda$, $0 < \lambda < 1$; it is said to be B-summable to S if $e^{-(x+y)} A(x, y) \to S$ as $x \to \infty$, $y \to \infty$. If in the above definitions we replace $A(x, y)$ by $a(x, y)$ the resulting summability methods are called, respectively, the B_λ' and B' methods. The authors announce a set of results which are mostly the two-dimensional analogues of known results on Borel summability [see, for example, Hardy, *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25] and typical of which is the following: If the double series converges to S and if

$$\left| \sum_{k=0}^{\infty} A_{ik} \frac{y^k}{k!} \right| \leq M_i e^{(1+\lambda')y}, \quad \left| \sum_{i=0}^{\infty} A_{ik} \frac{x^i}{i!} \right| \leq N_k e^{(1+\lambda')x},$$

where M_i, N_k, and $\lambda' < 1$ are positive numbers independent of x and y, then the double series is B_λ-summable to S if $\lambda > \lambda'$. They also announce a necessary and sufficient condition for the equivalence of the two methods B_λ and B_λ'. *M. S. Ramanujan* (Ann Arbor, Mich.)

APPROXIMATIONS AND EXPANSIONS

See also 8137, 8244, 8250, 8294.

8257:

Тиман, А. Ф. [Timan, A. F.]. ★Теория приближения функций действительного переменного [Theory of approximation of functions of a real variable]. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 624 pp. 23.05 r.

This book provides a rather detailed account of the literature of the theory of approximation, including some of the most recent results and most specialized problems. The basic classical results are not neglected, but the principal part of the book amounts to a coherent survey of the extremely large number of special results that have appeared, particularly in Russia, over the past two decades. Chapter I: Weierstrass' theorem. This includes approximation in the mean, weighted polynomial approximation on the whole real axis, and approximation by entire functions of exponential type. Chapter II: Best approximation. Chapter III: Some compact classes of functions and their structural characterization. This chapter includes properties of various moduli of continuity, properties of various classes of analytic and harmonic functions, quasi-analytic classes, and properties of conjugate functions (in the sense of harmonic analysis). Chapter IV.: Some properties of algebraic polynomials and

transcendental entire functions of exponential type. This chapter deals with interpolation formulas, the Paley-Wiener theorem, extremal properties and inequalities for polynomials and entire functions. Chapter V: Direct theorems of the constructive theory of functions. These are theorems in which the degree of approximation is deduced from structural properties of the function (continuity, differentiability, etc.). Chapter VI: Converse theorems. Constructive characteristics of some classes of functions. Here the structural properties are deduced from the degree of approximation. Chapter VII: Further theorems on the connection between best approximation of functions and their structural properties. Chapter VIII: Linear processes of approximation of functions by polynomials and some estimates connected with them. Here we find theorems on approximations by various summation methods and phenomena associated with the Lebesgue constants. There is a 25 page bibliography containing some 600 items. *R. P. Boas, Jr.* (Evanston, Ill.)

8258:

Il'in, V. P.; Solonnikov, V. A. Some properties of differentiable functions of many variables. Dokl. Akad. Nauk SSSR **136** (1961), 538–541 (Russian); translated as Soviet Math. Dokl. **2**, 95–98.

The author considers functions defined on domains in Euclidean space. He states without proof a number of results on approximation by means of smooth functions in various norms, and on the extension of functions to larger domains with preservation of norm and smoothness properties. Other theorems deal with the compactness of certain function classes. The results are too complicated to state in detail. The work is related to that of E. Gagliardo [Rend. Sem. Mat. Univ. Padova **26** (1956), 148–167; MR 18, 878], S. M. Nikol'skiĭ [Izv. Akad. Nauk SSSR **23** (1959), 677–696; MR **22** #1765], O. V. Besov [Dokl. Akad. Nauk SSSR **126** (1959), 1163–1165; MR **21** #5890]. *A. L. Shields* (New York)

8259:

Butzer, P. L. Fourier-transform methods in the theory of approximation. Arch. Rational Mech. Anal. **5**, 390–415 (1960).

The author continues his earlier work on saturation problems in approximation theory. The principal result of the paper is the following. Assume that $k(u)\geqq 0$ for $-\infty<u<\infty$, $\int_{-\infty}^{\infty} k(u)du=1$, there exists a measure ψ on $-\infty<u<\infty$ with $\int_{-\infty}^{\infty} d\psi(u)\neq 0$, and a constant $c>0$ such that $1-k^{\wedge}(v)=|v|^c\psi^{\wedge}(v)$, where $k^{\wedge}(v)$ is the Fourier transform of $k(u)$ and $\psi^{\wedge}(v)$ is the Fourier-Stieltjes transform of $\psi(u)$. Writing $f_r(u)=\int_{-\infty}^{\infty} f(u-t)rk(rt)dt$ it is shown that under the above assumptions, if $f(u)\in L^p(-\infty,\infty)$, $1<p\leqq 2$, then the following conditions are equivalent: (a) $\|f-f_r\|=O(r^{-c})$ as $r\to\infty$; (b) $|v|^c f^{\wedge}(v)\in\mathscr{F}^p$. Here $\mathscr{F}^p$ is the class of Fourier transforms of functions in $L^p(-\infty,\infty)$. The theorem is also true for $p=1$ with $\mathscr{F}^1$ replaced by $\mathscr{F}\mathscr{S}$, the class of Fourier-Stieltjes transforms. *I. I. Hirschman, Jr.* (Erlenbach)

8260:

Butzer, Paul L.; König, Heinz. An application of Fourier-Stieltjes transforms in approximation theory. Arch. Rational Mech. Anal. **5**, 416–419 (1960).

This paper continues the paper reviewed above [#8259] in the case $p=1$. Let the conditions on $k(u)$ be as above. However, assume now only that if $\psi^{\wedge}(v)$ is defined as $|v|^{-c}[1-k^{\wedge}(v)]$ for $v\neq 0$ then $\lim_{v\to 0}\psi^{\wedge}(v)$ exists and is not 0. ($\psi^{\wedge}(v)$ need not be a Fourier-Stieltjes transform.) It is shown that the implication (a)$\Rightarrow$(b) is still valid. On the other hand, if the implication (b)$\Rightarrow$(a) holds, then $\psi^{\wedge}(v)$ must be a Fourier-Stieltjes transform. *I. I. Hirschman, Jr.* (Erlenbach)

8261:

Havinson, S. Ya. On a class of extremal problems for polynomials. Dokl. Akad. Nauk SSSR **130** (1960), 997–1000 (Russian); translated as Soviet Math. Dokl. **1**, 137–140.

Here "polynomial" means a linear combination $\sum_1^n \lambda_\nu x_\nu$ of linearly independent elements x_ν of a locally convex linear topological space E. Let $p(x)$ be a continuous symmetric convex functional on E; let E_n be an n-dimensional linear space of points λ, with a locally convex topology; let $p_1(\lambda)$ be a continuous symmetric convex functional on E_n. The author states the equivalence of the problems: (I) find

$$\alpha=\inf_{(\lambda)}\Big[p\Big(y-\sum_{\nu=1}^{n}\lambda_\nu x_\nu\Big)+p_1(\lambda_1,\cdots,\lambda_n)\Big];$$

(II) find $\sup|f(y)|$ for all continuous linear functionals $f\in E^*$ such that $|f(x)|\leqq p(x)$, $x\in E$, and $|\sum_{\nu=1}^n\lambda_\nu f(x_\nu)|\leqq p_1(\lambda)$ for all $\lambda\in E_n$. Several concrete instances are mentioned. Extremal polynomials and extremal functionals are characterized in terms of each other. Let Q be a compact set and consider problems I and II for the space $C(Q)$. Then there is an r-point subset Q_r of Q ($r\leqq n+1$ in the real case, $r\leqq 2n+1$ in the complex case), such that the two problems have the same solutions for $C(Q)$ and for $C(Q_r)$. Chebyshev's theorem on points of maximum deviation is extended to the general case, but the number r of points is not necessarily $n+1$ in the real case unless p_1 is replaced by δp_1 with a sufficiently small δ. By way of illustration, the author states that he can solve explicitly, for small ε, the problem of finding $\sup|\int_{-1}^1 t^n\,dg|$ with $\int|dg|\leqq 1$, $|\int_{-1}^1 t^\nu dg|\leqq\varepsilon_\nu$, $\nu=1,\cdots,n-1$. *R. P. Boas, Jr.* (Evanston, Ill.)

8262:

Leont'ev, A. F. Completeness of certain systems of polynomials in domains of the complex plane. Dokl. Akad. Nauk SSSR **126** (1959), 939–942. (Russian)

Completeness of a set of polynomials in a region D means that every function analytic in D can be approximated, uniformly on compact subsets of D, by finite linear combinations of the given polynomials. Let $\{\Phi_n(z)\}$ be the Faber polynomials for a simply connected region G_∞ containing the point at infinity, and let $z\to\Phi(z)$ map the region conformally onto the exterior of a circle Γ about the origin in such a way that $\Phi(z)\sim z$ for large z. If L_1 is a curve from Γ to infinity and if L_2 is obtained from it by rotation through an angle φ, the region bounded by L_1, L_2, and an arc of Γ is called a curvilinear sector of angle φ. If (λ_n) is a sequence of positive integers of density τ, the Faber polynomials $\{\Phi_{\lambda_n}(z)\}$ are complete in any subregion D of G_∞ which is mapped by $z\to\Phi(z)$ into a curvilinear sector of angle $2\pi\tau$. Similarly, if the sequence (λ_n) has density τ, the Jacobi polynomials $\{P^{(\alpha,\beta)}{}_{\lambda_n}(z)\}$ are complete

in any region D obtained by the map $z \to (z+z^{-1})/2$ from a curvilinear sector of angle $2\pi\tau$ outside the unit disk.

L. de Branges (Bryn Mawr, Pa.)

8263:

Ahiezer, N. I. On polynomials orthogonal on a circular arc. Dokl. Akad. Nauk SSSR **130** (1960), 247–250 (Russian); translated as Soviet Math. Dokl. **1**, 31–34.

Let $t(\theta)$ be positive on $\alpha \leq \theta \leq 2\pi-\alpha$, and either continuous or with modulus of continuity $O(\log n)^{-1-\delta}$; the author finds the asymptotic form of the polynomials that are orthogonal on the arc $\alpha \leq \theta \leq 2\pi-\alpha$ with the weight function $w(\theta)=t(\theta)\sin\frac{1}{2}\theta\{\cos^2\frac{1}{2}\alpha-\cos^2\frac{1}{2}\theta\}^{-1/2}$. Since practically all previous investigations of polynomials orthogonal on the circle have been confined to those whose weight functions do not even vanish on a set of positive measure, the author's work is a new departure in its field.

R. P. Boas, Jr. (Evanston, Ill.)

8264:

Il'in, V. P. On the convergence of sequences of functions in some function spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 406–409.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 192–195 [MR **19**, 136].

8265:

Витушкин, А. Г. [Vituškin, A. G.]. ★Оценка сложности задачи табулирования [Estimation of the complexity of the tabulation problem]. Sovremennye Problemy Matematiki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 228 pp. 7.75 r.

In spite of the title of the book, it deals with a problem of approximation; we begin by outlining it. Let F be a set of continuous functions defined on a compact metric space G; let $\Phi(x,y)=\Phi_{\varepsilon,p}^{k,q}(x,y)$, $x\in G$, $y\in R^p$ be an ε-approximation algorithm (see below) for F, which we assume independent of f. This means that for each $f\in F$ there is a point $\bar{y}$ of the p-dimensional space R^p such that $|f(x)-\Phi_{\varepsilon,p}^{k,q}(x,\bar{y})|<\varepsilon$, $x\in G$. If $k=q=1$, Φ is linear: $\Phi_\varepsilon(x,y)=y_1f_1(x)+\cdots+y_pf_p(x)$. In this simplest case, Φ is an ε-approximation algorithm exactly if each $f\in F$ is ε-approximable by a linear combination of the given functions $f_1,\cdots,f_p$. In the general case, for each $x\in G$, $\Phi(x,y)$ is a piecewise rational function of degree k and order q. This means that R^p can be decomposed into parts γ_i by level surfaces of a polynomial of degree q in $y=(y_1,\cdots,y_p)$ (i.e., of degree q in each of the y_i) so that on each γ_i, $\Phi(x,y)$ is a quotient of two polynomials of degree k in y; the denominator must not vanish on γ_i. On the boundary of γ_i, Φ need not be uniquely defined. Now we formulate the main problem. What are the lower bounds for p,k,q for an ε-algorithm if F and $\varepsilon>0$ are given?

The author first develops the following tools. The entropy $H_\varepsilon(F)$ of a compact metric space F is the logarithm of the minimal number $n_\varepsilon(F)$ of the elements of a cover of F by subsets of diameter not exceeding 2ε. The rate of increase of $H_\varepsilon(F)$ for $\varepsilon\to 0$ is a measure of the massiveness of F. Chapters I–III are devoted to computations of the entropies of several important sets of functions [see also Kolmogorov and Tihomirov, Uspehi Mat. Nauk **14** (1959) no. 2 (86), 3–86; MR **22** #2890]. Another tool is the variations of sets in R^n, studied in chapter IV. If a set e is a subset of an n-dimensional cube J, if τ_k is one of the k-dimensional sides of J, and if $\beta_{n-k}(y)$ is the $n-k$-dimensional plane perpendicular to τ_k which contains $y\in\tau_k$, then the function of y which is equal to the number of components of the set $J\cap\beta_{n-k}(y)$ which do not intersect the boundary of J is a k-dimensional multiplicity of e. The average of all these over all τ_k of J is the k-dimensional variation $v_k^J(e)$ of e with respect to J. General theorems of this chapter describe the function $n_\varepsilon(e)$ of a set e, or the complement $J-e$, in terms of variations of e. Chapter V deals with properties of p-dimensional piecewise rational surfaces $e_p^{k,q}$ in R^n, given by equations $y_i=r_i(z)$, $i=1,\cdots,n$, $z\in R^p$ with piecewise rational functions r_i with parameters p,k,q. Results of Petrovskiĭ and Oleĭnik [Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 389–402; Amer. Math. Soc. Transl. no. 70 (1952); MR **11**, 613; **13**, 978] about level surfaces of polynomials allow an estimation of the variations of the $e_p^{k,q}$. Thus general theorems of chapter IV can be applied. We quote as examples: (1) Let e be a subset of the unit cube J in R^n with the distance to the boundary of J not less than 3ε. Assume further that a surface $e_p^{k,q}$ approximates e within at most ε. Then $n_{3\varepsilon}(e)\leq 2^n(p+1)^2[(q+1)(2k+q+1)/\varepsilon]^p$. (2) Let $e=e_p^{k,q}\subset R^n$; then $J-e$ contains an n-dimensional cube with side of length $d\geq\{3+[6^n(p+1)^2(q+1)^p(2k+q+1)^p]^{n-p}\}^{-1}$.

To obtain his main results, the author correlates the notions of entropy and variation by making special assumptions about the class of functions F. In chapter VI he calls F "easily approximable" if functions $\delta(\varepsilon)\geq 2\varepsilon$, $\delta(\varepsilon)\downarrow 0$ for $\varepsilon\to 0$ and $\nu(\varepsilon)$ exist such that for each $\varepsilon>0$ one can find a subset G_ν of G consisting of $\nu=\nu(\varepsilon)$ points for which the restriction F_ν of F to G_ν satisfies $\nu(\varepsilon)/H_{\delta(\varepsilon)}(F)\to 0$ for $\varepsilon\to 0$ and $H_\varepsilon(F_\nu)\geq H_{\delta(\varepsilon)}(F)$. Different classes of analytic functions fall into this category. In this case, for an ε-algorithm, $p\log[(q+1)(k+q+1)/\varepsilon]\geq H_\varepsilon(F)(1+o(1))$. Classes F "of type C" in chapter VII have the following property: for each $\varepsilon>0$, G contains a subset G_m of $m=m(\varepsilon)\geq AH_{B\varepsilon}(F)$ points so that the restriction of F to G_m contains an m-dimensional cube of side length 2ε; theorems of type (2) give $p\log[(q+1)(k+1)]\geq C_1H_{C_2\varepsilon}(F)$. Classes of differentiable functions with derivatives satisfying a Lipschitz condition are of this type.

{The reviewer concludes with some observations. The book seems to be a first attempt to deal with the degree of non-linear approximation; for linear algorithms, more precise results can be obtained by direct methods [Lorentz and Tihomirov, reviews immediately below]. The problem is of great theoretical interest; for a practical computor, the results of the present book seem not to favor the use of non-linear approximations; in the estimates p and k appear roughly combined as $p\log(k+1)$, so that it seems more profitable to increase p than k.}

G. G. Lorentz (Syracuse, N.Y.)

8266:

Lorentz, G. G. Lower bounds for the degree of approximation. Trans. Amer. Math. Soc. **97** (1960), 25–34.

Let X be a Banach space of real or complex valued functions f defined on a compact metric space A. Let $G=\{g_1,\cdots g_n,\cdots\}$, $g_n\in X$. Then

$$E_n{}^G(f)=\inf_{a_i}\|f-\sum_{i=1}^n a_ig_i\|$$

is the degree of approximation (d.a.) of f by G. $E_n{}^G(\mathfrak{A})=\sup E_n{}^G(f)$, $f\in\mathfrak{A}$ is the d.a. of a class $\mathfrak{A}\subset X$ and $D_n(\mathfrak{A})=\inf_G E_n{}^G(\mathfrak{A})$ is the optimal d.a. of $\mathfrak{A}$. The author gives a

simple method which permits finding estimates from below of $D_n(\mathfrak{A})$ for several important classes of $\mathfrak{A}$.

The characteristic of a compact family $\mathfrak{A}$, $\chi_n(\mathfrak{A})$, is the largest number $\delta \geqq 0$ with the following property. There exist $n+1$ points of A, $x_0, \cdots, x_n$, such that for each distribution of signs ε_k there exists $f \in \mathfrak{A}$ with $\operatorname{sign} f(x_k) = \varepsilon_k$, $|f(x_k)| \geqq \delta$, $k=0, 1, \cdots, n$. Using this definition, the author proves: $D_n(\mathfrak{A}) \geqq \chi_n(\mathfrak{A})$ for each compact family $\mathfrak{A}$ (theorem 1). Let $\mathfrak{A}$ be compact and convex, $0 \in \mathfrak{A}$; then for each sequence $\varepsilon_n \to 0$ and each G there is a function $f_0 \in \mathfrak{A}$ such that for infinitely many n, $E_n{}^G(f_0) \geqq \varepsilon_n \chi_n(\mathfrak{A})$ (theorem 2). He also presents complete proofs for results (1) and (2) announced in Bull. Amer. Math. Soc. **66** (1960), 124–125 [MR **22** #2829] (theorems 3, 5). With an appropriate definition of the characteristic $\delta_n{}^p$ in L^1-norm, he proves the result corresponding to theorem 1: For each subset $\mathfrak{A} \subset L^1(A)$ one has $D_n(\mathfrak{A}) \geqq p\delta_n{}^p(\mathfrak{A})$ in the L^1-norm (theorem 4). As an application of this result, one obtains for the class $\mathfrak{A}$ of analytic functions for the L^1-norm $D_n(\mathfrak{A}) \geqq Cn^{-1}\rho^{-n}$ (theorem 6). *M. Tomić* (Belgrade)

8267:

Tihomirov, V. M. On n-dimensional diameters of certain functional classes. Dokl. Akad. Nauk SSSR **130** (1960), 734–737 (Russian); translated as Soviet Math. Dokl. **1**, 94–97.

Let F be a subset of the Banach space R, and L_n an arbitrary n-dimensional subspace of R. If $\rho(x, L_n)$ denotes the distance from the point x to L_n, then $\delta(F, L_n) = \sup \rho(x, L_n)$ for all $x \in F$ is called the deflection of F from L_n. The infimum of the deflection, taken over all L_n, is called the n-dimensional diameter of F, and denoted by $d_n(F)$. The following theorem is proved: If U is the unit ball in R, and if $F_{n+1} = F \cap L_{n+1}$ and $U_{n+1} = U \cap L_{n+1}$ satisfy $\alpha U_{n+1} \subset F_{n+1}$ for some L_{n+1}, then $d_n(F) \geqq \alpha$. By means of the theorem the $2n$-dimensional diameters of several classes F of 2π-periodic continuous functions, equal to zero in the mean, are computed with respect to the uniform norm. One example is the class F_r of real $f(x)$ of the stated kind for which $\|f^{(r)}\|_\infty \leqq 1$. It is proved that $d_{2n}(F_r) = \delta(F_r, T_{2n})$ where T_{2n} is the subspace of trigonometric polynomials of order n without constant term. Furthermore, $d_{2n}(F_r) = A_r(n+1)^{-r}$ where A_r is explicitly computed. *A. C. Zaanen* (Pasadena, Calif.)

8268:

Tihomirov, V. M. Diameters of sets in functional spaces and the theory of best approximations. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 81–120 (Russian); translated as Russian Math. Surveys **15** (1960), no. 3, 75–111.

This expository paper deals with the n-dimensional diameters or widths of sets of functions with the emphasis on their exact determination and the case of the C-space norm. Let R be a Banach space, F, G subsets of R, then $\delta(F, G)$ denotes the least upper bound of distances of points $x \in F$ to G. If M_n denotes an arbitrary n-dimensional linear manifold in R, then $d_n(F) = \inf_{M_n} \delta(F, M_n)$. For central-symmetric F, one can restrict in this definition the M_n to n-dimensional linear subspaces of R. For the estimates of the widths from below the author follows his own paper [preceding review], for the estimations from above the results of Ahiezer and Kreĭn [Dokl. Akad. Nauk SSSR **15** (1937), 107–112], Favard [Bull. Sci. Math. (2) **61** (1937), 209–224, 243–256] and others. In particular, for some sets of periodic functions, exact values of the $d_n(F)$ are obtained. {Theorem 1 (p. 84) about convex sets, which the author uses for this purpose, has also been given by Gohberg and Kreĭn [Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118; MR **20** #3459].} *G. G. Lorentz* (Syracuse, N.Y.)

8269:

Rice, John R. The characterization of best nonlinear Tchebycheff approximations. Trans. Amer. Math. Soc. **96** (1960), 322–340.

Let $F = F(a, x)$ be a continuous real function on $P \times [0, 1]$, where P is a subset of the n-dimensional euclidean space E_n, and assume that if $a \neq a^*$, then $F(a, x) \neq F(a^*, x)$ for some $x \in [0, 1]$. For a continuous function $f(x)$, any $F(a^*, x)$ such that

$$\max_{0 \leqq x \leqq 1} |F(a^*, x) - f(x)| \leqq \max_{0 \leqq x \leqq 1} |F(a, x) - f(x)|$$

for all $a \in P$ is called a best approximation of $f(x)$. F is said to have the property NS if, for every continuous function $f(x)$, the alternance n times of $\max |F(a^*, x) - f(x)|$ is necessary and sufficient for $F(a^*, x)$ to be a best approximation of $f(x)$. ($\operatorname{Max} |F(a^*, x) - F(x)|$ is said to alternate n times if there are $n+1$ points of $[0, 1]$ in which $F(a^*, x) - f(x)$ attains its maximum and with opposite signs for consecutive points.) The purpose of the present paper is to give, under various assumptions on the function F, some characterizations of the property NS. In theorem 1 it is assumed only that F is continuous, and there is given a rather complicated characterization of property NS. In theorem 3 it is assumed that F is "closed", in the following sense: P is arcwise connected and $\lim_{k\to\infty} F(a_k, x) = G(x)$ with $(x, G(x)) \in R$ (where $R = \{(x, F(a, x)) | x \in [0, 1], a \in P\}$) implies that there exists an $a_0 \in P$ such that $\lim_{k\to\infty} a_k = a_0$. Theorem 3 asserts that if F is closed, then local unisolvence is a necessary and sufficient condition for F to have property NS. (F is said to be locally unisolvent if given $0 \leqq x_1 < \cdots < x_n \leqq 1$, $a^* \in P$ and $\varepsilon > 0$, there is a $\delta(a^*, \varepsilon, x_1, \cdots, x_n) > 0$ such that whenever $|y_j - F(a^*, x_j)| < \delta$, there is a unique $a \in P$ with $F(a, x_j) = y_j$, $j = 1, \cdots, n$, $\max |F(a, x) - F(a^*, x)| < \varepsilon$.) Moreover, F is said to be n-point closed if P is connected and if $\lim_{k\to\infty} F(a_k, x_j) = y_j$, $j = 1, \cdots, n$ with $(x_j, y_j) \in R$ implies the existence of an $a_0 \in P$ such that $F(a_0, x_j) = y_j$. F is said to be unisolvent if given a set $\{(x_j, y_j) | 0 \leqq x_1 < \cdots < x_n \leqq 1, (x_j, y_j) \in R\}$, there is a unique $a \in P$ such that $F(a, x_j) = y_j$, $j = 1, \cdots, n$. It is proved (theorem 4) that if F is n-point closed, then unisolvence is a necessary and sufficient condition for F to have property NS [the sufficiency of this condition has been proved, even under the more general hypothesis that F is only continuous, e.g., by L. Tornheim, Trans. Amer. Math. Soc. **69** (1950), 457–467; MR **12**, 395; and M. I. Morozov, Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952), 75–100; MR **13**, 728; the main achievement of theorem 4 is the necessity part]. Finally, the notions of property NS and unisolvence are extended to include some more general situations, and a final theorem (theorem 5) relates these generalized concepts. *Ivan Singer* (Bucharest)

8270:

Duncan, Cecil E. On the asymptotic behavior of trigonometric sums. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 261–264, 369–380.

(I) G. N. Watson [Philos. Mag. (6) **31** (1916), 111–118] determined for large positive ω the asymptotic behavior of the sum $\sum_{0<n<\omega} f(n)$, where $f(x) = \csc \pi\omega/x$, by means of the Laurent expansion of $(e^t-1)^{-1}$ according to powers of t. By using the similar expansion of $e^{at}(e^t-1)^{-1}$ the author determines the asymptotic behavior of sums of a more general type, namely $\sum_{n=a+1}^{b} g(n)$, where $g(x) = x^T \csc^{(t)} \pi(x/\omega+\gamma)$ or $x^T \cot^{(t)} \pi(x/\omega+\gamma)$; here t and T are integers $\geqq 0$ and $\csc^{(t)} u$ is the tth derivative of $\csc u$ with respect to u. (II) The author treats the same sums with the sum formula of Euler written in an appropriate form. (III) Euler's sum formula cannot be applied in the usual way on the sum $\sum_{0<n<\omega} f(n)$, where $f(x) = \csc \pi\omega/x$, since $f(x)$ is not integrable from 0 to ω. However the integral of $f(x)$ from 0 to ω has a finite part according to Hadamard. By using this finite part and by assigning to the function $f(x)$ and its derivatives suitable values at $x=0$ and at $x=\omega$, the author shows that the sum formula of Euler yields immediately the result obtained by Watson concerning this sum. This device extends considerably the domain of applicability of Euler's sum formula in the theory of asymptotic expansions.

J. G. van der Corput (Berkeley, Calif.)

FOURIER ANALYSIS

See also 8109, 8131, 8235, 8259, 8260.

8271:

Hsu, L. C. The uniform approximation to the Lipschitz class of functions by a kind of trigonometrical polynomials. Math. Student **26** (1958), 155–160.

For $0<\alpha\leqq 1$, let $\mathrm{Lip}_M\,\alpha$ denote the class of functions F satisfying $|F(x')-F(x'')| \leqq M|x'-x''|^\alpha$. Let μ and n be positive integers, μ odd, and let

$$V_n^\mu(F;x) = K_n^{-1}\int_{-\pi}^{\pi}\{\cos\tfrac{1}{2}(t-x)\cos\tfrac{1}{2}\mu(t-x)\}^n F(t)\,dt,$$

where $K_n = 4\int_0^{\pi/2}(\cos\theta\cos\mu\theta)^n\,d\theta$. The author shows that if $U_n^\mu(\alpha) = \sup\,(\max_x |V_n^\mu(F;x)-F(x)|)$, where the supremum is taken over all $F\in\mathrm{Lip}_1\,\alpha$, then for large even n

$$U_n^\mu(\alpha) = \Gamma(\tfrac{1}{2}(1+\alpha))\pi^{-1/2}(8/(1+\mu^2)n)^{\alpha/2}\{1+o(1)\}.$$

For $\mu=1$, this result is due to de la Valleé Poussin [Acad. Roy. Belg. Bull. **1908**, 193–254]; in this special case n need not be even.

P. G. Rooney (Toronto)

8272:

Videnskiĭ, V. S. Extremal estimates for the derivative of a trigonometric polynomial on an interval shorter than its period. Dokl. Akad. Nauk SSSR **130** (1960), 13–16 (Russian); translated as Soviet Math. Dokl. **1**, 5–8.

Let

$$t_n(\theta) = \cos 2n\cos^{-1}\{\sin\tfrac{1}{2}\theta/\sin\tfrac{1}{2}\omega\},$$

$$u_n(\theta) = \sin 2n\cos^{-1}\{\sin\tfrac{1}{2}\theta/\sin\tfrac{1}{2}\omega\}.$$

Let $s_n(\theta)$ be a trigonometric polynomial of order n such that $|s_n(\theta)|\leqq 1$, $-\omega\leqq\theta\leqq\omega$, $0<\omega<\pi$; the author proves that, for $-\omega<\theta<\omega$,

$$|s_n'(\theta)| \leqq |t_n'(\theta)+iu_n'(\theta)| = n\cos\tfrac{1}{2}\theta\,[\sin^2\tfrac{1}{2}\omega-\sin^2\tfrac{1}{2}\theta]^{-1/2},$$

and, for $n>\frac{1}{2}[3\tan^2\frac{1}{2}\omega+1]^{1/2}$,

$$|s_n'(\theta)| \leqq t_n'(\omega) = 2n^2\cot\tfrac{1}{2}\omega,\quad -\omega\leqq\theta\leqq\omega,$$

with equality only for $s_n(\theta)=\gamma t_n(\theta)$, $|\gamma|=1$, and only at the roots of $t_n(\theta)$ or at $\pm\omega$, respectively.

R. P. Boas, Jr. (Evanston, Ill.)

8273:

Goes, G. Charakterisierung von Fourierkoeffizienten mit einem Summierbarkeitsfaktorentheorem und Multiplikatoren. Studia Math. **19** (1960), 133–148.

The author continues his investigations on spaces of Fourier coefficients [cf. Math. Z. **70** (1958/59), 345–371; Math. Ann. **137** (1959), 371–384; MR **21** #3711, 7392]. For the notations we refer to the reviews of these papers. Let $f=\sum_1^\infty a_n\cos nx$, and let $\Delta^k a_n$ denote the sequence of the kth differences of the sequence a_n for an arbitrary $k\geqq 0$. Theorem 1: If $\sum_{n=1}^\infty (n+1)^k|\Delta^{k+1}a_n|<\infty$ for some $k>0$, then $f\in dV$ if and only if $a_n=O(1)$, and $f\in L$ if and only if $a_n=o(1)$. Theorem 2: If $\sum|\Delta a_n|<\infty$, then $f\in(dV)^{**}$, and hence $f\in V^*$. Theorem 3: If $\sum|\Delta a_n|<\infty$ and $a_n=o(1)$, then $f\in L^{**}$. The proofs make use of a theorem of Schur-Bosanquet. There are some similar results for $\sum a_n\sin nx$. The paper also contains a number of theorems on multiplicator sequences, and the final section is devoted to additional remarks, partly of bibliographical nature, concerning the earlier work on this subject by the author and others.

A. C. Zaanen (Pasadena, Calif.)

8274:

Miracle, Chester L. The Gibbs phenomenon for Taylor means and for $[F, d_n]$ means. Canad. J. Math. **12** (1960), 660–673.

It is proved here that both the Taylor and $[F, d_n]$ transforms of a Fourier series exhibit the Gibbs phenomenon at any point where the generating function (assumed to satisfy the Dirichlet conditions) has a discontinuity and that, moreover, the Gibbs set in each case coincides with the Gibbs set associated with the partial sums of the Fourier series at the point in question. These results are established by considering the function $\varphi(x)$ of period 2π which equals $-\pi/2$ for $-\pi<x<0$ and which equals $\pi/2$ for $0<x<\pi$, with $\varphi(-\pi)=\varphi(0)=\varphi(\pi)=0$. Detailed definitions are provided, both for the Gibbs phenomenon and for the two summation methods considered. The $[F, d_n]$ method is a generalization of both the Euler and Lototsky methods.

L. Lorch (Edmonton, Alta.)

8275:

Cross, George. The expression of trigonometrical series in Fourier form. Canad. J. Math. **12** (1960), 694–698.

The author shows the following theorem. If the trigonometric series $\sum c_n e^{int}$ is bounded (C,k) in $(-\pi,\pi)$, where k is a fixed integer, then the coefficients c_n can be written in Fourier form using the $C_{k+1}P$-integral. This is a straightforward generalization of Burkill's result [J. London Math. Soc. **11** (1936), 43–48].

G. Sunouchi (New Haven, Conn.)

8276:

Yang, Chao-Hui. On the integrability of functions defined by cosine series with monotone decreasing coefficients. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 73–80.

Let $f(x) \sim \frac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n \cos nx$, $\lambda_n \downarrow 0$. It is well-known that neither $f(x)$ nor its conjugate, denoted here by $g(x)$, need belong to $L(0, \pi)$ [cf. Zygmund, *Trigonometric series*, 2d. ed., vol. I, Cambridge Univ. Press, New York, 1959; MR **21** #6498; pp. 184, 185, 378]. A number of authors have established conditions under which $\eta(x)f(x)$ and $\eta(x)g(x)$ are in $L(0, \pi)$, particularly for $\eta(x) = x^{-r}$, $r \geq 0$. In the present paper, $\eta(x)$ is assumed only to be nonnegative and in $L(0, \pi)$, and results involving $f(x)$ are obtained. Typical is theorem 1:

If $\sum_1^\infty \lambda_n \int_0^{1/n} \eta(x)dx < \infty$, then $\eta(x)f(x) \in L(0, \pi)$. Analogous results involving $g(x)$ are given by Peyerimhoff [Arch. Math. **9** (1958), 75–81; MR **21** #2156].

L. Lorch (Edmonton, Alta.)

8277:

Zygmund, A. On the preservation of classes of functions. J. Math. Mech. **8** (1959), 889–895; erratum, **9** (1960), 663.

On dit qu'une suite $\{\lambda_n\}$ $(n = \cdots, -1, 0, 1, \cdots)$ est du type (P, Q) si $\sum c_n \lambda_n e^{inx}$ représente une fonction de la classe P chaque fois que $\sum c_n e^{inx}$ représente une fonction de la classe Q. On s'intéresse aux classes Λ_α, λ_α, Λ_*, λ_*, $\Lambda_\alpha{}^p$, $\lambda_\alpha{}^p$, $\Lambda_*{}^p$, $\lambda_*{}^p$ $(0 < \alpha < 1;\ 1 \leq p \leq \infty)$ définies par l'auteur dans son livre [*Trigonometric series*, 2d. ed., vol. I, Cambridge Univ. Press, New York, 1959; MR **21** #6498; pp. 42–43, 45]. On sait [loc. cit., p. 178] que si

$$(1) \qquad \sum{}' (in)^{-1}\lambda_n e^{inx}$$

est la série de Fourier d'une fonction à variation bornée, $\{\lambda_n\}$ est de chacun des types $(\Lambda_\alpha, \Lambda_\alpha)$, $(\lambda_\alpha, \lambda_\alpha)$, (Λ_*, Λ_*), (λ_*, λ_*); l'auteur montre que la condition nécessaire et suffisante pour qu'il en soit ainsi est que (1) représente une fonction $\in \Lambda_*{}^1$. Il établit que les types $(\Lambda_\alpha{}^p, \Lambda_\beta{}^p)$ et $(\Lambda_\alpha{}^{p'}, \Lambda_\beta{}^{p'})$ sont identiques dès que $1/p + 1/p' = 1$.

J.-P. Kahane (Montpellier)

8278:

Korenblyum, B. I. Harmonic analysis of functions of rapid growth. Amer. Math. Soc. Transl. (2) **16** (1960), 416–417.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 201–203 [MR **18**, 892].

8279:

Gurarii, V. P. A generalization of the Fourier transform and the Paley-Wiener theorem. Dokl. Akad. Nauk SSSR **130** (1960), 959–962 (Russian); translated as Soviet Math. Dokl. **1**, 102–105.

The author extends Ahiezer's theory [Dokl. Akad. Nauk SSSR **96** (1954), 889–892; MR **16**, 242] of Fourier transforms on the space $L_\varphi{}^2$ of functions with scalar product $\int_{-\infty}^\infty \{f\bar{g}/\varphi\}dx$. He supposes that $\varphi(z)$ is an entire function of exponential type 0, nonnegative on the real axis, with zeros satisfying $\sum |\Im(1/a_k)| < \infty$. Then there is another entire function $\omega(z)$ of exponential type 0 such that $\varphi(x) = |\omega(x)|^2$, with zeros z_k in the upper half plane. Let

$$\omega_k(z) = (2\pi i)^{-1/2}(z_k - \bar{z}_k)^{1/2}\omega(z)\prod_{j=1}^{k-1}(z - \bar{z}_j)\Big/\prod_{j=1}^{k}(z - z_j).$$

Then $f(x) \in L_\varphi{}^2$ if and only if

$$f(x) = \sum_{k=1}^\infty a_k\omega_k(x) + (2\pi)^{-1/2}\omega(x)\int_0^\infty e^{ixt}h(t)\,dt + (2\pi)^{-1/2}\bar{\omega}(x)\int_{-\infty}^0 e^{ixt}h(t)\,dt$$

with $h \in L^2$; and $\|f\|^2_{L^2_\varphi} = \sum |a_k|^2 + \|h\|^2_{L^2}$. The author then gives the analogue of the Paley-Wiener theorem and a result showing that the conditions imposed on φ cannot be weakened if the Fourier reciprocity is to hold. He also states that if $L_\varphi{}^2$ contains all polynomials then the system of polynomials is complete in the space of entire functions of exponential type zero that belong to $L_\varphi{}^2$, if and only if

$$\sum P_n(z)P_n(w) = \frac{1}{2\pi i}\,\frac{\omega(z)\bar{\omega}(w) - \bar{\omega}(z)\omega(w)}{z - w},$$

where $\{P_n\}$ is an orthonormal set in $L_\varphi{}^2$.

R. P. Boas, Jr. (Evanston, Ill.)

8280:

Duffin, R. J.; Shaffer, D. H. Asymptotic expansion of double Fourier transforms. Duke Math. J. **27** (1960), 581–596.

Soit $S(r)$ une fonction de classe C^∞ de $r = [x^2 + y^2]^{1/2}$, égale à zéro pour $r > r_2$ et à 1 pour $r < r_1$; si on pose $F(x, y) = r^q x^\alpha y^\beta S(r)$ (α, β entiers ≥ 0, $q + \alpha + \beta > -2$, $q \neq -2, -4, \cdots$) la transformée de Fourier f de F admet un développement asymptotique de la forme

$$f(a, b) \sim L(q)\left(\frac{\partial}{i\partial a}\right)^\alpha\left(\frac{\partial}{i\partial b}\right)^\beta [a^2 + b^2]^{-1-q/2}.$$

Les cas limites $(q = -2, -4, \cdots;\ \alpha + \beta + q = -2)$ sont ensuite étudiés. Les auteurs en déduisent le développement asymptotique de la fonction de Green de la théorie discrète du potentiel.

J. Lelong (Paris)

8281:

Bredihina, E. A. Some problems concerning the approximation of almost periodic functions with a bounded spectrum. Dokl. Akad. Nauk SSSR **131** (1960), 721–724 (Russian); translated as Soviet Math. Dokl. **1**, 306–309.

The author considers uniformly almost periodic (u.a.p.) functions $f(x) \sim \sum_{-\infty}^\infty A_k e^{i\Lambda_k x}$ such that $\Lambda_0 = 0$, $\Lambda_{k+1} < \Lambda_k$ $(k > 0)$, $\lim_{k\to\infty} \Lambda_k = 0$, $\Lambda_{-k} = -\Lambda_k$, $A_0 = 0$. She relates the following quantities:

$$\Omega_f(N) = \sup_{|T| \geq N}\sup_x \left|(1/T)\int_0^T f(x+t)\,dt\right| \quad (N > 0),$$

$$R_\varepsilon(f) = \sup_x \Big|f(x) - \sum_{|\Lambda_k| > \varepsilon} A_k e^{i\Lambda_k x}\Big|,$$

$$e_\varepsilon(f) = \inf_{c_k}\sup_x \Big|f(x) - \sum_{|\Lambda_k| > \varepsilon} c_k e^{i\Lambda_k x}\Big|,$$

$$E_\varepsilon(f) = \inf_{F \in Q_\varepsilon}\sup_x |f(x) - F(x)|,$$

where Q_ε is the class of u.a.p. functions with exponents λ_k such that $|\lambda_k| > \varepsilon$ for all k. The two principal results are (1) $e_\varepsilon(f) \leq C\Omega_f(1/\varepsilon)$ and (2) $R_\varepsilon(f) \leq 2\phi(\varepsilon, \eta)E_\varepsilon(f)$, where

$$\phi(\varepsilon, \eta) = 1 + 2/\pi + N(\eta) - N(\varepsilon) + (1/\pi)\ln(\varepsilon + \eta)/(\varepsilon - \eta)$$

and $N(\varepsilon) = \sum_{\Lambda_k \geq \varepsilon} 1$. From these follow various results concerning the convergence of the Fourier series of f; for

instance, if there is an integer $m \geq 1$ and a real number $\vartheta > 1$ such that $\Lambda_n/\Lambda_{n+m} \geq \vartheta$ (all n), then the Fourier series of f converges uniformly. This paper follows one in which the author obtained similar results for u.a.p. functions with Fourier exponents Λ_k such that $\lim_{k\to\infty} \Lambda_k = \infty$ [same Dokl. **123** (1958), 219–222; MR **21** #260].

H. Burkill (Sheffield)

8282:

Laha, R. G. On a property of positive-definite functions. Bull. Amer. Math. Soc. **66** (1960), 388–391.

Positive definite functions of a real variable have the following incipient property of analyticity. If a positive definite function $f(t)$ is such that for a sequence $t_k \downarrow 0$, $f(t_k) = \psi(t_k)$, $f(-t_k) = \psi(-t_k)$, where $\psi(t)$ is analytic in a complex neighborhood of the origin, then $f(t) \equiv \psi(t)$. If $\psi(t)$ is assumed to be positive definite and $\psi(t) = \psi(-t)$, the theorem had been proven by Yu. V. Linnik [Teor. Veroyatnost. i Primenen. **1** (1956), 466–478; MR **19** 693].

S. Bochner (Princeton, N.J.)

8283:

McMillin, Kenneth M. Abel summability of the double series successively derived from the conjugate of the double Fourier series. Bull. Calcutta Math. Soc. **51** (1959), 179–185.

In a previous paper [Tôhoku Math. J. (2) **8** (1956), 183–187; MR **20** #4147] the author extended a result of M. L. Misra [Duke Math. J. **14** (1947), 167–177; MR **8**, 577] regarding the Abel summability of the successively derived Fourier series from one variable to two variables. In the present paper the analogous problem for the successively derived conjugate double Fourier series is studied.

R. Mohanty (Cuttack)

8284:

Malliavin, Paul. Calcul symbolique et sous-algèbres de $L_1(G)$. I, II. Bull. Soc. Math. France **87** (1959), 181–186, 187–190.

Soit G un groupe abélien discret infini, G' le groupe dual de G, A l'algèbre des fonctions réelles, transformées de Fourier des fonctions $\in L^1(G)$. On dit que F opère sur $a \in A$ si la fonction composée $F(a) \in A$. Par la méthode qui lui a permis d'établir l'impossibilité de la synthèse spectrale [C. R. Acad. Sci. Paris **248** (1959), 1756–1759, 2155–2157; MR **21** #5854a, b], l'auteur démontre le théorème suivant: soit M_n une suite positive telle que les suites (1) $\{\log(M_{n+1}/M_n)\}$ et (2) $\{n^{-1}\log(M_n/n!)\}$ soient croissantes, $I = [-l, l]$, et $C(M_n, I)$ l'ensemble des fonctions f telles que $\sup_{n,x\in I} |f^{(n)}(x)/M_n|^{1/n} < \infty$; on suppose que $C(M_n, I)$ est non-quasi-analytique, c'est-à-dire ne contient aucune fonction $\neq 0$ à support dans I; alors il existe $a \in A$, norme $\|a\|$ aussi voisine qu'on le veut de l, telle que toute fonction F opérant sur a appartient à $C(M_n, I)$. Sans hypothèse sur la suite (2), on obtient seulement $m(x)F(x) \in C(M_n, I)$, où $m(x)dx$ est l'image par a de la mesure de Haar sur G'. Le théorème suivant, plus facile, donne une condition suffisante pour que F opère sur a: si les coefficients de Fourier γ_k de a satisfont $\sum |\gamma_k| \log |\gamma_k| > -\infty$, il existe une suite M_n comme cidessus telle que, si I contient à son intérieur les valeurs prises par a, toute fonction $F \in C(M_n, I)$ opère sur a. C'est un raffinement des anciens résultats de Marcinkiewicz [Mathematica (Cluj) **16** (1940), 66–73; MR **1**, 329]. L'auteur indique une extension de ces théorèmes aux fonctions F de plusieurs variables, opérant sur plusieurs éléments de A.

J.-P. Kahane (Montpellier)

8285:

Kahane, J. P. Propriétés locales des fonctions à séries de Fourier aléatoires. Studia Math. **19** (1960), 1–25.

Consider the series (1) $\sum_{n=1}^{\infty} R_n(\cos nt + \phi_n)$ where R_n and ϕ_n are mutually independent random variables for which the expected value of $\cos \phi_n$ and $\sin \phi_n$ is 0 for every n. Denote by $F(t)$ the series represented by (1).

The author obtains conditions (in some cases necessary and sufficient ones) which insure that with probability 1 $F(t)$ should satisfy the following conditions: I. uniform convergence of (1); II. continuity of $F(t)$; III. $F(t) \in L^\infty$; IV. $F(t)$ belongs to Lip α. In this short review only some of the simplest and most striking of his results can be stated in detail.

Assume that (1) is of the form

$$\sum_{n=1}^{\infty} a_n(\phi_n \cos nt + \varphi_n \sin nt)$$

where $a_n \geq 0$ and ϕ_n and φ_n are independent random variables having Gauss distribution and mean 0, variance 1; put $s_i = (2\sum_{n=2^i+1}^{2^{i+1}} a_n^2)^{1/2}$. Assume that the sequence s_i is decreasing. Then the necessary and sufficient condition that with probability 1 any of the conditions I, II, III should hold is that $\sum s_i$ converges.

Put $\alpha^* = \liminf_{i\to\infty} (-\log s_i)/(i \log 2)$. If $\alpha < \alpha^*$ then with probability 1 $F(t)$ is in Lip α, and if $\alpha > \alpha^*$ the probability that $F(t)$ is in Lip α is 0.

An interesting result of a more special character is the following: Let $a_n = n^{-3/2}(\log n)^\gamma$. If $\gamma < -1$, $F(t)$ is with probability 1 everywhere continously differentiable. If $\gamma > 0$, $F(t)$ is with probability 1 nowhere differentiable. If $-1 \leq \gamma < -\frac{1}{2}$, $F(t)$ (with probability 1) is not in Lip 1 but is the primitive function of a function in L^2 (this is no longer the case if $\gamma \geq -\frac{1}{2}$).

[Compare Salem and Zygmund, Acta Math. **91** (1954), 245–301; MR **16**, 467.]

P. Erdös (Budapest)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 8115, B9374.

8286:

Everitt, W. N. On a generalization of Bessel functions and a resulting class of Fourier kernels. Quart. J. Math. Oxford Ser. (2) **10** (1959), 270–279.

The existence of certain Fourier kernels, such as $\pi^{-1/2}(e^{-x} + \cos x - \sin x)$ which are simple combinations of exponential and circular functions, was noted by the reviewer [same J. **1** (1950), 191–193; MR **12**, 175]. The present paper shows that these kernels satisfy differential equations of the form $y^{(2k)} + (-1)^{k+1}y = 0$ for some positive integer k, and generalizations of these kernels are found which are analogous to the extension of the Fourier sine and cosine kernels to the Hankel kernels. Some remarks are made at the end about the connexion between these results, eigenfunction expansions, and the separability of some partial differential equations.

A. P. Guinand (Saskatoon, Sask.)

8287:

Nudel'man, A. A. Limit values of integrals

$$\int_a^{\xi\pm 0} \Omega(t)\, d\sigma(t)$$

under Markov conditions. Dokl. Akad. Nauk SSSR **131** (1960), 1253–1256 (Russian); translated as Soviet Math. Dokl. **1**, 419–422.

For basic ideas and notation, one must refer to M. G. Kreĭn's paper on the Čebyšev-Markov theory of limiting values of integrals [Uspehi Mat. Nauk **6** (1951), no. 4 (44), 3–120; Amer. Math. Soc. Transl. (2) **12** (1959), 1–122; MR **13**, 445; **22** #3947a]. In the present paper, the author studies canonical representations of index $n+2$ with respect to variation of the generalized moments. He also investigates the range of variation of the integrals in the title as the first $n+1$ moments of $d\sigma(t)$ vary over a parallelepiped, the results being of possible use for estimating the integrals when the moments are known only approximately.

In his study of canonical representations of index $n+2$, the author is able to determine the nature of the changes in the masses of such representations as one of the first $n+1$ moments increases. The exact nature of the behavior depends on the order of the moment, whether n is odd or even, and how the masses of the representation are located with respect to the mass points of the upper and lower principal representations.

D. S. Greenstein (Evanston, Ill.)

8288:

Verma, C. B. L. Two theorems on Meijer-Transform. Bull. Calcutta Math. Soc. **51** (1959), 171–178.

The Meijer transform mentioned in the title was developed by Meijer in Nederl. Akad. Wetensch. Proc. **44** (1941), 727–737, 831–839 [MR **3**, 38, 109]. It involves Whittaker functions.

The two theorems are designed to show that if a function $\phi(p)$ can be obtained from a function $f(x)$ by the successive applications of two Meijer transforms and then a Laplace transform, then $\phi(p)$ may be determined directly from $f(x)$ by means of a formula involving Meijer's G-functions.

J. L. Griffith (Kensington)

8289:

Narain, Roop. Some properties of generalized Laplace transform. III. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 85–93.

[For part I see Riv. Mat. Univ. Parma **8** (1957), 283–306; MR **21** #5122.]

The transform considered in this paper is given by

$$\Phi(s; k, m) = W[f(t); k, m]$$
$$= s\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)f(t)\, dt.$$

Two chain theorems are obtained. In the first a chain of $f_i(t)$ is used to determine a function which has the image $\Phi(s^2; (2k-1)2^{-n}+\frac{1}{2}, m2^{-n+1})$. The second theorem contains a chain of Φ_i which can be used to find

$$W[f(t^{2^{n-1}}); k, m].$$

A corollary of this last theorem is found in a previous paper by the same author [Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956/57), 429–432; MR **20** #1174]. The remark made by the reviewer in that review holds here. It appears that the restriction $R(s) \geqq s_0 > 0$ should be replaced by a statement implying that all equations are valid for all s such that $R(s) > 0$.

Some applications to G-functions are given.

J. L. Griffith (Kensington)

8290:

Narain, Roop. Some properties of generalized Laplace transform. IV. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 35–41.

The generalized Laplace transform mentioned in the title is Varma's transform

$$W[f(t); k, m] = s\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)f(t)\, dt,$$

where $W_{k,m}(x)$ is the Whittaker function.

A formula is found which expresses $W[t^{\nu-1}f(1/t); \lambda, \mu]$ in terms of $W[f(t); k, m]$ by means of an integral involving Meijer's G-function.

A number of particular cases of the formula are given. This paper continues the work reviewed above [#8289]. [See also Math. Z. **68** (1957), 272–281; MR **21** #3732.]

J. L. Griffith (Kensington)

8291:

Norman, Edward. A discrete analogue of the Weierstrass transform. Proc. Amer. Math. Soc. **11** (1960), 596–604.

The author has considered the inversion of a class of discrete convolution transforms $f(n) = \sum_{-\infty}^{\infty} I_{n-m}g(m)$ analogous to the Weierstrass transform

$$f(x) = \int_{-\infty}^{\infty} \exp\left[-(x-y)^2/4\right]g(y)\, dy.$$

(There are printing mistakes, e.g., 2.13 instead of 1.3 in line 11 p. 597 and W^m instead of $W^{|m|}$ in line 8, p. 601.) No particular functions $f(n)$ have been considered as an application of the theory.

D. P. Banerjee (Tirupati)

8292:

Kirillov, A. A. A problem of I. M. Gel'fand. Dokl. Akad. Nauk SSSR **137** (1961), 276–277 (Russian); translated as Soviet Math. Dokl. **2**, 268–269.

Let K be a complex curve in C^n given by the equation $x = \varphi(\lambda)$. The complex lines which intersect K form a manifold M of n complex dimensions. For every indefinitely differentiable and rapidly decreasing function f on C^n we define the transform $\check{f}$ on M by

$$\check{f}(\alpha, \lambda) = \int f(\varphi(\lambda)+t\alpha)\, dt\, d\bar{t}$$

(α is a vector, indicating the direction of a line in M). The author proves that $\check{f}$ is uniquely determined by f if and only if K intersects almost all hyperplanes in C^n. In case K intersects almost all hyperplanes in exactly l points, the author proves the inversion formula

$$f(x) = \frac{-1}{4l\pi^2}\int D\check{f}(x-\varphi(\lambda), \lambda)\, d\lambda\, d\bar{\lambda},$$

where

$$D = \sum\left(\frac{\partial\varphi_i}{\partial\lambda}\frac{\partial\bar{\varphi}_j}{\partial\bar{\lambda}} - \frac{\partial\varphi_i}{\partial\bar{\lambda}}\frac{\partial\bar{\varphi}_j}{\partial\lambda}\right)\frac{\partial^2}{\partial\alpha_i\partial\bar{\alpha}_j}.$$

The main tool of the proofs is the Fourier transform. Analogous results hold also in the case of a real curve in a real space. *A. Korányi* (Berkeley, Calif.)

8293:

LePage, Wilbur R. ★Complex variables and the Laplace transform for engineers. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1961. xvii + 475 pp. $12.50.

This book develops the theory of Fourier and Laplace transforms together with the basic theory of functions of a complex variable. The presentation of motivating examples and exercises from electrical engineering situations presupposes some knowledge of circuit theory and elementary Laplace transform technique on the part of the reader. At the start the treatment makes only modest demands on the reader's mathematical knowledge or maturity. But in later chapters the reader is exposed to a proof of the Cauchy-Goursat theorem, the author's viewpoint on impulse functions, and a partially intuitive discussion of conditions for the inversion of order in repeated integrals with infinite limits. The author has done well in his selection of topics of use to engineers. A few of his definitions, as on pages 31 and 32, and some proofs as on page 34 lack precision. The use of σ, ω for x, y is unfortunate, since to save the student one translation at a late stage of the subject, it requires a little extra mental effort each time a fact about the xy-plane is applied. The interpretations of mathematical ideas may be helpful to some readers, but will seem clumsy to others. Thus this book can be recommended as desirable to only a limited category of engineering students. *P. Franklin* (Cambridge, Mass.)

8294:

König, Heinz. Einige Eigenschaften der Fourier-Stieltjes-Transformation. Arch. Math. **11** (1960), 352–365.

The author develops various results concerning the cosine-Stieltjes transform $P(t)=\int_0^\infty \cos tu \; d\Phi(u)$ where $\Phi(\cdot)$ is a non-negative finite measure on the Borel subsets of $0\leq u<\infty$. Let $\phi(u)=\int_u^\infty d\Phi(v)$. It is shown that for $0<\lambda\leq 2$ the conditions $\lim_{t\to 0+}[P(0)-P(t)]t^{-\lambda}=B/\Gamma(\lambda+1)$ and $\lim_{u\to\infty} u^\lambda\phi(u)=(2B/\pi\lambda)\sin(\pi\lambda/2)$ are equivalent. Also conditions are obtained which are sufficient to insure that $[P(0)-P(t)]t^{-\lambda}$ is again a cosine-Stieltjes transform. Here the principal result is that if $u^\lambda\phi(u)$ is of bounded variation on $0\leq u<\infty$ then $[P(0)-P(t)]t^{-\lambda}$ is the cosine-Stieltjes transform of a not necessarily non-negative measure. If however $u^\lambda\phi(u)$ is non-decreasing and bounded the measure is non-negative. Also in the special case $\lambda=2$ the measure, when it exists, is necessarily non-negative. These results have been developed for application, elsewhere, to the theory of approximation and saturation classes.

I. I. Hirschman, Jr. (Erlenbach)

8295:

Erdélyi, A. ★Lectures on Mikusiński's theory of operational calculus and generalized functions. California Institute of Technology, Pasadena, Calif., 1959. iii + 137 pp. (mimeographed)

These lecture notes give a beautiful account of Mikusiński's operational calculus. However, not all the material can be found in Mikusiński's book [*Operational calculus*, English edition by Pergamon Press, New York, 1959; MR 21 #4333]. There are notable additions and refinements by the author, especially in the more sophisticated parts of the chapters preparing for and dealing with partial differential equations (see below). Although the treatment is at a somewhat more advanced level than in Mikusiński's book, a student with a good background in advanced calculus should be able to read, understand, enjoy and apply the material in these lecture notes. Indeed, they should form an excellent basis for a (graduate) course in operational calculus. {Such a course should no longer be based exclusively on Laplace transform theory; Mikusiński's theory is more natural, less restrictive, and gives a good deal of additional insight into such phenomena, for example, as traveling and reflected waves, where these arise as solutions of boundary value problems.}

Chapter I (Introduction) contains a brief sketch of Heaviside calculus, Laplace transforms, the delta function and distributions. Chapter II (The algebra of convolution quotients) introduces the field of convolution quotients and discusses the differential operator s, the integral operator h and other operators as elements of the field. There are applications to ordinary linear differential equations with constant coefficients and to integral equations of convolution type. Chapter III (The analysis of convolution quotients) deals with sequences and series of convolution quotients, defines operator functions, in particular exponential functions, and gives applications to linear difference equations with constant coefficients. Chapter IV (Differential equations involving operator functions) is a preparation for the chapters on partial differential equations. Here are careful discussions of the operator differential equations $z''(x)-s^kz(x)=0$ or $=f(x)$, for $k=2$, 1. Various theorems give sufficient conditions under which these equations have solutions $z(x)=\{z(x,t)\}$ of class C^2. Chapters V (The one-dimensional wave equation) and VI (The one-dimensional diffusion equation) discuss initial and boundary value problems for these equations both for infinite and finite intervals. Conditions are given under which solutions of class C^2 exist, and there is a detailed discussion of generalized solutions of the boundary value problems. *J. Korevaar* (Madison, Wis.)

8296:

Mikusiński, J. Remarks on the algebraic derivative in the Operational Calculus. Studia Math. **19** (1960), 187–192.

This paper is a continuation of the work in chapter VII of the author's book *Operational calculus* [Pergamon, New York, 1959; MR 21 #4333].

In that chapter the operation D is defined by $D\{f(t)\}=\{-tf(t)\}$ and suitably extended to apply to quotients of such operators. D is the analogue of d/ds of Laplace transform theory.

Five new properties of D are found. This first is: $Dx=0$ implies that $x=$ a number. It is shown that this theorem can be formulated as: the condition $\int_0^t (t-2\tau)f(t-\tau)\times g(\tau)d\tau=0$ for $0\leq t\leq\infty$ is sufficient and necessary for the functions f and g to be linearly independent for $0\leq t<\infty$.

The fourth property states that: if $Dx=wx$ is solvable, the solution is determined up to a numerical factor.

In the final section some comment is made on the analytic continuation of the Laplace transform of e^{wt^2}. (There is an "s" missing from the formula (11) on p. 192.)

J. L. Griffith (Kensington)

8297:

McCully, Joseph. The Laguerre transform. SIAM Rev. **2** (1960), 185–191.

The basic operational calculus of the Laguerre integral transformation $T\{F\}=\int_0^\infty e^{-x}L_n(x)F(x)dx$ is presented, where $L_n(x)$ is the Laguerre polynomial of degree n. In particular if $f(n)$ is the transform of $F(x)$, $f=T\{F\}$, the differential form $L[F]=e^x[xe^{-x}F'(x)]'$ has the transform $-nf(n)$. A convolution property of the transformation, giving the inverse transform of the product $f(n)g(n)$ as an iterated integral of the object functions $F(x)$ and $G(x)$, is a major contribution of the paper. The author gives sufficient conditions for the validity of the operational properties. He also lists a short table of transforms. An illustrative boundary value problem in a partial differential equation containing the differential operator L is solved by applying the operational calculus and using the convolution property to simplify the result.

R. V. Churchill (Ann Arbor, Mich.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

8298:

Gol'dennšteĭn, L. S.; Gohberg, I. C. On a multi-dimensional integral equation on a half-space whose kernel is a function of the difference of the arguments, and on a discrete analogue of this equation. Dokl. Akad. Nauk SSSR **131** (1960), 9–12 (Russian); translated as Soviet Math. Dokl. **1**, 173–176.

Results of M. G. Kreĭn [Uspehi Mat. Nauk **13** (1958), no. 5 (83), 3–120; MR **21** #1507] for the equation (1) $\varphi(t)-\int_0^\infty k(t-s)\varphi(s)ds=f(t)$ $(0\leqq t<\infty)$ and its discrete analog (2) $\sum_{k=0}^\infty a_{j-k}\xi_k=\eta_j$ $(j=0,1,2,\cdots)$ are extended to the multi-dimensional versions of these equations, where, for example, in (1) $t-s$ may now be any point of the real n-dimensional Euclidean space E while s, t are confined to the half-space E^+: $0\leqq t_1<\infty$, $-\infty<t_j<\infty$ $(j=2,3,\cdots,n)$. Where formerly for $n=1$ the basic facts of the theory of equations (1), (2) were found to be essentially the same, essential differences appear when $n>1$. Statements of the results are too involved for a brief review.

J. F. Heyda (Cincinnati, Ohio)

8299:

Sakalyuk, K. D. Abel's generalized integral equation. Dokl. Akad. Nauk SSSR **131** (1960), 748–751 (Russian); translated as Soviet Math. Dokl. **1**, 332–335.

The author gives in closed form the solution of the equation

$$(1)\qquad c(x)\int_a^x\frac{\varphi(t)\,dt}{(x-t)^\alpha}+d(x)\int_x^b\frac{\varphi(t)\,dt}{(t-x)^\alpha}=f(x)\qquad(0<\alpha<1)$$

by reducing it to the solution of Abel's integral equation. Here $c(x)$ and $d(x)$, which are defined on the interval $[a,b]$, are assumed not to vanish simultaneously and to have derivatives satisfying Hölder's condition. The solution is sought in a class of functions of a particular type. The author gives the needed formulas without deriving them.

H. P. Thielman (Oxnard, Calif.)

8300:

Mihlin, S. G. Remarks on the solution of multi-dimensional singular integral equations. Dokl. Akad. Nauk SSSR **131** (1960), 1019–1021 (Russian); translated as Soviet Math. Dokl. **1**, 375–378.

This is an extension and simplification of the results obtained by the author in an earlier paper in same Dokl. **117** (1957), 28–31 [MR **20** #1897].

H. P. Thielman (Oxnard, Calif.)

8301:

Levin, J. J.; Nohel, J. A. On a system of integro-differential equations occurring in reactor dynamics. J. Math. Mech. **9** (1960), 347–368.

The authors study the system:

$$(1)\qquad \frac{du}{dt}=-\int_{-\infty}^{\infty}\alpha(x)T(x,t)\,dx,\quad aT_t=bT_{xx}+\eta(x)u\qquad(0\leqq t<\infty);$$

subject to

$$(2)\qquad u(0)=u_0,\quad T(x,0)=f(x)\quad(-\infty<x<\infty),$$

which arises in a linear theory of medium reactor kinetics. The fundamental solution of the diffusion equation is employed to formally reduce the problem (1), (2) to a standard Volterra integral equation whose solution is then shown to rigorously determine a solution of (1), (2). Under stated conditions this solution is proved unique. The asymptotic behavior (as $t\to\infty$) of a solution is derived by application of a Tauberian theorem due to Von Stachó. Finally it is shown that as $b\to0+$ the solution converges uniformly in $0\leqq t\leqq T_1$, for any $T_1<\infty$ and each x to the explicit solution of (1), (2) in which $b=0$. It is also shown that the main results are valid under weaker hypotheses than those employed.

H. B. Keller (New York)

FUNCTIONAL ANALYSIS

See also 8107, 8129, 8133, 8181, 8182, 8216, 8227, 8261, 8294.

8302:

Dunford, Nelson; Schwartz, Jacob T. ★Linear Operators. I. General Theory. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958. xiv + 858 pp. $23.00.

This is a comprehensive account of the modern theory of linear operators, mainly in Banach spaces, and of applications of the theory of such operators to other parts of mathematics.

The present volume I under review contains the topological theory of spaces and operators, and the spectral theory of "arbitrary" operators and some applications; the second volume will contain the spectral theory of completely reducible operators and further applications, e.g., applications to differential operators and partial differential equations.

With exception of the first introductory one, each of the eight chapters of volume I concludes with a section called "Notes and remarks", consisting partly of valuable sketches of the historical development and partly of

supplementary material. The authors state that the great majority of these sections are due to R. G. Bartle.

In a book of such large scope as the present one there is always the danger that the reader gets lost in the maze of definitions, notations, lemmas and theorems. The authors make every effort to combat this danger by using (among others) the following devices: (i) each chapter is preceded by a short outline of, and often by a motivation for, the basic concepts dealt with in that chapter; the same is true for many of the sections into which each chapter is divided; (ii) there is an extensive subject index; (iii) there is an alphabetic index of notations; (iv) a great number of results concerning various spaces and operators are collected in tabulated form with reference to the respective sections in the book or to the literature; (v) there is an "interdependence table" depicting graphically the relations between the various sections of the book.

In spite of all this the book is not all easy reading; sometimes the tools designed to make things less complicated are complicated themselves, such as the graphs mentioned above. But perhaps it is unfair to expect of a work like the present one which has nearly the character of a handbook the degree of readability which we find, e.g., in the well-known book by Riesz-Nagy.

The following (by no means complete) survey of the book's content will show the wealth of material presented.

Chapter I, entitled "Preliminary concepts", is divided into three parts: part A deals with set-theoretic notions and theorems, including partial ordering, Hausdorff's "maximality theorem", Zorn's lemma, well-ordering; part B deals with topological notions and theorems like Hausdorff, normal, compact, metric spaces, Tietze's extension theorem, Baire's category theorem, sequences and generalized sequences, the Moore-Smith theorem; filters are introduced very briefly and used to prove Tychonoff's theorem; part C deals with algebraic notions like groups, linear spaces, Boolean rings, Stone's theorem on the representation of such rings, determinants.

Chapter II is entitled "Three basic principles of analysis". The three principles are: (1) The principle of uniform boundedness (or equi-continuity) for certain families of linear mappings; (2) the interior mapping principle, asserting the openness of the image of an open set under a linear continuous map of an "F-space" onto another such space; (3) The Hahn-Banach theorem. In connection with this latter theorem the study of weak convergence and reflexivity (of spaces) is started, topics pursued in a more detailed manner in chapter V.

Chapter III deals with the theory of integration based on the theory of set functions (which may be real or vector-valued, i.e., having values in a Banach space) and on measures, which are defined as special scalar-valued set functions. The functions to be integrated may be real or vector-valued. The first three sections develop the theory based on finitely additive set functions. From section 4 on, the theory is based on countably additive functions defined on a σ-field. Section 4 contains the basic definitions and the Hahn and Lebesgue decomposition theorems for measures. Section 5 deals with the extension of measures via the Caratheodory outer measure and the Hahn extension theorem. Regular measures, and the measures named after Borel, Radon, and Lebesgue, are defined. Section 6 treats the integration theory based on the measure theory of the two previous sections. The classical convergence theorems (like the Lebesgue dominated convergence theorem) are in this section, which also contains definitions and theorems on convergence in measure and other topics, e.g., the completeness of L_p spaces. Sections 7 and 8 deal with measure spaces and with relative measures μ_1 obtained by restricting the domain of a given measure μ. In section 10 the Radon-Nikodym theorem is proved. Section 11 is devoted to product measures. Here the Fubini theorem is proved and a complement to it due to Tonelli. Also theorems due to Jessen treating the case of products of a countable number of measures are contained in this section. Section 12 is concerned with the differentiation of set functions in finite-dimensional Euclidean spaces. As a tool the Vitali covering theorem is proved. In section 14 various classical theorems of the theory of functions of a complex variable (including the Weierstrass preparation theorem) and extensions of such theorems are assembled for later use.

In chapter IV, entitled "Special spaces", 26 linear spaces are listed, all of them Banach spaces or F-spaces. With the exception of two (Euclidean n-space and Hilbert space) they are all "concrete" spaces, i.e., spaces of functions (or equivalence classes of functions). Most of these spaces are treated in considerable detail. For example, the section on the space $C(S)$ of continuous functions on a normal space S contains among many other topics the Riesz representation theorem and the Stone-Weierstrass theorem (for S compact Hausdorff); the section on almost periodic functions contains Bochner's characterization of these functions in terms of their translates. All the 26 spaces are investigated with respect to 8 questions listed at the beginning of the chapter. To illustrate the type of questions asked we quote the following ones. (1) What is an analytical representation of the conjugate space X^* of the given space X? (4) Is X weakly complete? (5) Is X reflexive? (6) Which subsets of X are weakly sequentially compact? A summary of the results known concerning these questions is given in form of tables with reference to the place in the chapter where the problem is treated. The results obtained in a section on certain special spaces of set functions is used to extend the integration theory of chapter III to integrals with respect to vector-valued measures.

Chapter V is entitled "Convex sets and weak topologies". Section 1 introduces basic notions about convex sets (e.g., the idea of a support function) in general linear spaces, i.e., linear spaces with no topology; the section culminates in the basic separation theorem asserting the existence of a linear functional separating two given disjoint convex sets one of which has as an internal point. In section 2 the results of section 1 are applied and sharpened in the case that the underlying space is linear and topological. For example, the basic separation theorem is sharpened to assert the existence of a linear continuous separating functional. Section 3 brings definitions and fundamental properties of weak topologies. The theorem concerning the identity of strong and weak* closedness for convex sets in a locally convex linear topological space is contained in this section. In section 4 the study of weak topologies is continued, particularly with respect to compactness and reflexivity. Among the results proved here are the theorem of Alaoglu and the related theorem asserting that the weak compactness of the unit ball in a Banach space X is a necessary and sufficient condition for the reflexivity of X. Section 5 deals with the metrizability of Banach spaces X in their weak topologies

and with the so-called bounded topology for the conjugate space X^*. Section 8 contains the Eberlein-Šmul'yan theorem asserting the equivalence of the following three properties of a subset A of a Banach space X: (i) A is weakly sequentially compact; (ii) every countably infinite subset of A converges weakly to an element of X; (iii) the closure of A in the X^*-topology is X^*-compact. In the remaining part of the chapter the following topics are treated: extremal points (Kreĭn-Mil'man theorem), functionals tangent to convex sets, and fixed-point theorems (the Schauder-Tychonoff and Markov-Kakutani theorems).

Chapter VI takes up again the study of the space $B(X, Y)$ of linear continuous maps of a Banach space X into a Banach space Y started in chapter II. In section 1 various topologies of $B(X, Y)$ are defined, the main consideration being given to the uniform, the strong, and the weak operator topologies. In section 2 the notion of the operator $T^* \in B(Y^*, X^*)$ adjoint to the given operator $T \in B(X, Y)$ is introduced, and section 3 introduces the family of projections in $B(X, X)$. The next three sections deal with weakly compact operators, compact operators, and with the important case of operators whose range is closed. Section 7 deals with the representation in some analytic form of elements of $B(X, C(S))$ and elements of $B(C(S), X)$ where X is a Banach space and where $C(S)$ denotes the space of continuous functions on the compact Hausdorff space S. Section 8 deals with the analogous problem obtained if $C(S)$ is replaced by a Lebesgue space. References to the literature for the representation of operators in many other spaces and also for operators with specified properties (e.g., compact operators) are compiled in tabulated form. One more topic to be mentioned in this chapter is the proof of the M. Riesz convexity theorem by a method due to G. O. Thorin.

Chapter VII is entitled "General spectral theory", the word "general" indicating that it deals with "arbitrary" operators in contradistinction to completely reducible operators to be discussed in volume II. In this chapter starts the blending of the topological tools used in the previous chapters with function-theoretic and algebraic tools. The treatment is in the spirit of some of the senior author's earlier papers [see, e.g., Trans. Amer. Math. Soc. **54** (1943), 185–217; MR **5**, 39]. It is based on the Cauchy formula

$$(*) \qquad f(T) = (2\pi i)^{-1} \int f(\lambda)(\lambda I - T)^{-1}\, d\lambda.$$

The theory is first developed for operators in finite-dimensional spaces. In section 3 the study of bounded operators in a Banach space is started. After the basic definitions (spectrum, resolvent, etc.) are given, a number of classical results concerning the operational properties of the correspondence (defined by (*)) between a certain function $f(\lambda)$ defined on the spectrum of T and the operator $f(T)$ are proved: e.g., the homomorphism concerning addition and multiplication of functions $f(\lambda)$, $g(\lambda)$, the composition $(f \circ g)(\lambda)$, and the spectral mapping theorem. We also mention the theorem concerning the equality of the Riesz index and the order of a pole of the resolvent. Section 4 deals with compact linear operators and contains various generalizations to such operators (started by F. Riesz) of the classical spectral properties of Fredholm integral operators with non-singular kernels. Section 6 deals with perturbation theory following mainly the lines of Rellich's original investigation (1936); the main result is the extension of a theorem by Rellich. The principal topic treated in section 7 is the following "Tauberian" problem: let $f(\lambda)$, $f_1(\lambda)$, $f_2(\lambda)$, $\cdots$ be functions belonging to a certain well-defined family, and let $f(T)$, $f_1(T)$, $f_2(T)$, $\cdots$ be the operators corresponding to them by the formula (*) above; suppose that the operators $f(T)\cdot f_n(T)$ converge in some operator topology; find sufficient conditions that the operators $f_n(T)$ converge. A special case of this problem arises in ergodic theory. So far the theory has been restricted to bounded operators. Section 9 deals with unbounded, not everywhere defined, operators T. T is supposed to be closed, i.e., the graph of T is a closed set. Moreover, it is assumed that the resolvent set of T is not empty. This implies the existence of a scalar α such that $A = (T - \alpha T)^{-1}$ is bounded. Previous results may thus be applied to A and translated into results on T. It turns out that formula (*) again exhibits the basic correspondence if to its right member the term $f(\infty)I$ is added. The results of this section are due to A. E. Taylor [Acta Math. **84** (1951), 189–224; MR **12**, 717].

The last chapter of volume I deals with two topics: semigroups and ergodic theory. Section 1 centers about three theorems. The first one asserts that for an operator A to be the infinitesimal operator of a uniformly continuous semigroup of operators it is necessary and sufficient that A be bounded. The second one is the Hille-Yosida-Phillips theorem stating a necessary and sufficient condition for a closed densely defined operator to be the infinitesimal operator of a strongly continuous semigroup of operators. The third one gives a necessary and sufficient condition for the closed "perturbation operator" P that the operator $A + P$ be closed and the infinitesimal operator of a semigroup, provided A is the infinitesimal operator of a strongly continuous semigroup of bounded operators. Section 2 takes up the problem of assigning to the infinitesimal operator A of a strongly continuous semigroup of operators a bounded operator $f(A)$, where $f(\lambda)$ belongs to a certain class of functions defined on the spectrum of A. Since A is closed and in general unbounded this is connected with the problem treated in section 9 of chapter VII; but the class of functions $f(\lambda)$ considered in the present case is greater; it is the class of bilateral Laplace-Stieltjes transforms. The inversion problem for the operators of $f(A)$ is also considered. In general $[f(A)]^{-1}$ is unbounded. The inverse is therefore not given in "closed form" but as a limit of polynomials. The remainder of the chapter is devoted to ergodic theory. Section 4 is introductory; it discusses the connection with statistical mechanics, the Liouville theorem, the representation of measure-preserving transformations by unitary operators and Markov processes. Sections 5 and 6 deal, in the discrete case, with the mean and pointwise ergodic theorem. Section 7 deals with the continuous case. In section 8 sufficient conditions on a bounded operator T are obtained for $n^{-1} \sum_0^n T^j$ to converge in the uniform operator topology.

The preceding outline of the book's content has taken into account only the main body of the text. As already pointed out there is supplementary material in the sections entitled "Notes and remarks". In addition there are a great number of exercises, many of which contain an essential amount of additional material and of applications to "hard" analysis. Examples for this are the following: (1) on page 75ff. (chapter II) there is a connected series of exercises pertaining to summability methods for divergent

series; (2) section 14 of chapter IV consists of exercises concerned with the application of linear space methods to orthogonal series, including Fourier series, multiple Fourier series, Poisson summability, Tchebycheff polynomials, to name a few topics; (3) section 11 of chapter VI consists of exercises in which a great number of inequalities are to be derived using the Riesz convexity theorem of the preceding section.

A natural question to ask is how the book compares in intent and scope with other books on linear spaces. The authors discussed this question themselves (preface, p. vii); there is no need to repeat their discussion. The present book is an important contribution to the mathematical literature. The wealth of material and the extensive bibliography should make it a standard reference for the expert in the field. It should also prove valuable to the advanced graduate student seeking knowledge in one of the many subjects treated. The latter, however, should be warned that one important part of modern linear space theory is touched on only occasionally, namely, the theory of linear spaces which are not normed. If, e.g., he wants to be informed about the present state of the theory of linear convex topological spaces, he ought to turn to other sources. *E. H. Rothe* (Ann Arbor, Mich.)

8303:

Choquet, Gustave. Limites projectives d'ensembles convexes et éléments extrémaux. C. R. Acad. Sci. Paris **250** (1960), 2495–2497.

Let $\mathscr{V}$ be a topological vector space and X a part of $\mathscr{V}$. A 'slice' of X is a non-void intersection of X with an open affine half-space. An element $x \in X$ is a 'strong extremal point' of X if the set of all slices containing x is a fundamental system of x in X. Denote by $E_f(X)$ and $E(X)$ the set of strong extremal points and the set of extremal points of X, respectively. Many results concerning (especially) $E_f(X)$ are given, among them the following. (1) If $\mathscr{V}$ is separated and X is convex then $E_f(X) \subset E(X)$. There is a bounded closed convex part X of a Hilbert space such that $E_f(X) \neq E(X)$. (2) The set $E_f(X)$ is a G_δ in X if X has countable basis. (3) If $\mathscr{V}$ is locally convex and separated, X closed, and X' the smallest closed convex set containing X, then $E_f(X) = E_f(X')$. (4) Let $\mathscr{P}$ be the class of all cones C such that (a) $C \subset \mathscr{V}$, where $\mathscr{V}$ is a separated locally convex space which is the projective limit of a family $(\mathscr{V}_j)_{j \in J}$ of locally convex spaces, (b) $C = \mathscr{V} \cap \prod_{j \in J} C_j$, where, for each $j \in J$, $C_j \subset \mathscr{V}_j$ is a cone with compact basis. Let C be a convex cone in $\mathscr{V}$. Then $C \in \mathscr{P}$ if and only if $C \ni 0$, C contains no straight line passing through the origin, C is weakly complete and the polar C^0 spans the dual of $\mathscr{V}$. The class $\mathscr{P}$ is "stable under projective limit". Every $C \in \mathscr{P}$ is isomorphic to a vaguely closed cone of positive measures on a discrete space. For each $C \in \mathscr{P}$ we have $E_f(C) = E(C)$. (5) Let $\mathscr{D}$ be the class of all $C \in \mathscr{P}$ such that (a′) J is countable, (b′) for each $j \in J$, C_j has compact metrizable basis. If $C \in \mathscr{D}$, then C is metrizable and complete, $E_f(C) = E(C)$ is a G_δ in C, and the smallest closed convex set containing $E_f(C) = E(C)$ contains C.

C. Ionescu Tulcea (New Haven, Conn.)

8304:

Gordon, Hugh. Decomposition of linear functionals on Riesz spaces. Duke Math. J. **27** (1960), 597–606.

Let V be a Riesz space, i.e., an ordered linear space such that $\sup(f, g)$ and $\inf(f, g)$ exist for every $f, g \in V$. A linear functional F on V is a point functional if $|Ff| = F|f|$ for each $f \in V$. A relatively bounded linear functional F on V (i.e., difference of two positive linear functionals) is an atomic functional if there exists a set A of point functionals and a family $(\alpha_G)_{G \in A}$ of numbers such that $Ff = \sum_{G \in A} \alpha_G Gf$ for each $f \in V$. A relatively bounded linear functional F on V is a diffuse functional if for every $\varepsilon > 0$ and for every positive $f \in V$ there are pairwise disjoint relatively bounded functionals $F_1, \cdots, F_n$ such that $|F| = F_1 + \cdots + F_n$ and $F_i f < \varepsilon$ for each i. The author proves the following theorem: any relatively bounded linear functional on V may be expressed as the sum of an atomic functional and a diffuse functional. Suppose now that V is both a Riesz space and an algebra of real-valued functions on a set E and that V contains the constant functions. The following theorem is proved: every multiplicative linear functional is a point functional and every point functional F which is normalized so that $Fe = 1$ is multiplicative. Here e is the constant function 1.

N. Dinculeanu (Bucharest)

8305:

Amemiya, Ichiro; Andô, Tsuyoshi; Sasaki, Masahumi. Monotony and flatness of the norms by modulars. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 96–113 (1959).

For norms on linear lattices, two dual concepts, (uniform) monotone and (uniform) flat, are defined in H. Nakano, J. Fac. Sci. Imp. Univ. Tokyo Sect. I **4** (1942), 201–382 [MR **9**, 191]. For a modulared space, two norms, the first and second norms, are defined in H. Nakano, *Modulared semi-ordered linear spaces* [Maruzen, Tokyo, 1950; MR **12**, 420]. The authors obtain some conditions on a modular by which one of its norms become uniformly monotone or flat. *H. Nakano* (Kingston, Ont.)

8306:

Itô, Takashi. On conjugately similar transformations. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 125–152 (1959).

Let R be a reflexive linear lattice with the conjugate $\bar{R}$. In order that there exist a finite monotone complete modular on R with finite conjugate modular on $\bar{R}$, it is necessary and sufficient that there exist a one-to-one lattice-preserving correspondence T from R to $\bar{R}$, that is, $T(x \cup y) = Tx \cup Ty$, $T(x \cap y) = Tx \cap Ty$, $T0 = 0$. Such a correspondence T is called a conjugately similar transformation from R to $\bar{R}$ [H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420]. The author generalizes this theorem for general, not necessarily finite, modulars. A mapping T from a subset M of R into $\bar{R}$ is called a conjugately similar transformation on R, and M is called the domain of T, if: M consists of only positive elements; $M \ni a \geqq b \geqq 0$ implies $b \in M$; $a, b \in M$ implies $a \cup b \in M$; for any $0 \leqq a \in R$ there exists $\xi > 0$ for which $\xi a \in M$; $M \ni \xi a > 0$ for all $\xi > 0$ implies $T\xi_0 a > 0$ for some $\xi_0 > 0$; $M \ni a \geqq b \geqq 0$ implies $Ta \geqq Tb \geqq 0$; for any $a \in M$ and projection operator P on R we have $T(Pa) = (Ta)P$; and $M \ni a_\lambda \uparrow_{\lambda \in \Lambda} a$, $\sup_{\lambda \in \Lambda} (a_\lambda, Ta_\lambda) < +\infty$ implies $\alpha a \in M$ for $0 \leqq \alpha < 1$. For a conjugately similar transformation T with domain M, putting

$$m_T(a) = \int_0^1 (|a|, T\xi|a|)\, d\xi \quad (a \in R),$$

where $(|a|, T\xi|a|) = +\infty$ for $\xi|a| \notin M$, we obtain a monotone complete modular m_T on R. Conversely, for any monotone complete modular m on R, we can find T for which $m = m_T$. Thus every modular on R can be characterized by such a T. The author attempts, furthermore, classification of modulars by means of such conjugately similar transformations. *H. Nakano* (Kingston, Ont.)

8307:

James, Robert C. Separable conjugate spaces. Pacific J. Math. **10** (1960), 563–571.

A highly ingenious construction is given to prove the following. Let T be a Banach space with an orthogonal basis which is boundedly complete. Then there exists a Banach space B with a shrinking basis such that $B^{**} = Q(B) \oplus T_1$, where $Q(B)$ is the natural imbedding of B in B^{**} and T_1 is isometric with T. This result is a substantial generalization of the author's well-known construction for T = the real line. As an application, a construction of a Banach space B_n is given such that the nth conjugate space is the first nonseparable conjugate space of B_n.

V. Pták (Prague)

8308:

Luxemburg, W. A. J. A remark on R. R. Phelps' paper "Subreflexive normed linear spaces". Arch. Math. **11** (1960), 192–193.

Suppose E is a real normed linear space and V is a set of bounded linear functionals on E. If V is closed under scalar multiplication, then V is dense in E^* if and only if for every bounded closed convex set C in E and every x in $E-C$ there is an f in V such that $f(x) > \sup(f(y): y \text{ in } C)$. This generalizes a result due to R. R. Phelps [Arch. Math. **8** (1957), 444–450; **9** (1958), 439–440; MR **20** #6027; **21** #2175]. *D. C. Kleinecke* (Berkeley, Calif.)

8309:

Singer, Ivan. On Banach spaces reflexive with respect to a linear subspace of their conjugate space. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 448–462.

Let X be a real normed linear space, and let Y be a subspace of the conjugate space X^*. Let φ be the mapping from X into Y^* defined by $\varphi(x)(y) = y(x)$, $y \in Y$. Let S denote the unit ball of X, and let $\Sigma = \{x \in X | \sup_{y \in Y}(|y(x)|/\|y\|) \leq 1\}$. The space X is called Y-reflexive if φ is an isometric isomorphism of X onto Y^*. The author showed [J. London Math. Soc. **34** (1959), 320–324; MR **21** #5884] that X is Y-reflexive if and only if S is compact on the topology $\sigma(X, Y)$. He now shows that φ is one-to-one if and only if Σ is compact in $\sigma(X, Y)$, and uses this to obtain an alternate proof of the prior result. He shows that S is dense in Σ in $\sigma(X, Y)$, and that φ is an isomorphism of X onto Y^* if and only if Σ is strongly bounded. Among the other results noted is that X is isometrically isomorphic to a conjugate space if and only if it is Y-reflexive for some subspace Y of X^*.

P. Civin (Gainesville, Fla.)

8310:

Alexiewicz, A.; Semadeni, Z. Some properties of two-norm spaces and a characterization of reflexivity of Banach spaces. Studia Math. **19** (1960), 115–132.

This continues work of the authors in (I) Studia Math. **17** (1958), 121–140 [MR **20** #6644] and (II) ibid. **18** (1959), 275–293 [MR **22** #5878]. Here the two-norm space $(X, \|\ \|, \|\ \|^*)$ is called quasi-normal if there is a positive number K such that $\|x_n - x_0\|^* \to 0$ implies that $\|x_0\| \leq K \liminf_n \|x_n\|$; $K=1$ for a normal space, as in (I). Equivalent conditions to quasi-normality are given, one of which shows that if the space is quasi-normal, a norm $\|\ \|_1$ can be found such that $K\|x\|_1 \geq \|x\| \geq \|x\|_1$ and $(X, \|\ \|_1, \|\ \|^*)$ is normal; hence many properties of normal spaces carry over to quasi-normal spaces. Examples show that there need be neither a finest nor a coarsest $\|\ \|^*$ determining a given subspace Ξ_γ of Ξ. Reflexivity is analysed more carefully and conditions are given that $\Xi_\gamma = \Xi$ or $\Xi_\gamma = \Xi^*$. Theorem 3.3 shows that a Banach space $(X, \|\ \|)$ is reflexive if and only if for every $\|\ \|^*$ coarser than $\|\ \|$, the space Ξ^* is dense in $(\Xi, \|\ \|)$.

M. M. Day (Urbana, Ill.)

8311:

Fréchet, Maurice. L'espace des courbes est-il un espace de Banach? Bul. Inst. Politehn. Iași (N.S.) **5 (9)** (1959), no. 1–2, 31–34. (Russian and Romanian summaries)

The author suggests a method for attacking a problem that he subsequently solved [C. R. Acad. Sci. Paris **250** (1960), 2787–2790; MR **22** #4943].

C. W. Kohls (Rochester, N.Y.)

8312a:

Fréchet, Maurice. Exemples de semi-espaces de Banach. C. R. Acad. Sci. Paris **251** (1960), 1702–1703.

8312b:

Fréchet, Maurice. L'espace dont chaque élément est une courbe n'est qu'un semi-espace de Banach. C. R. Acad. Sci. Paris **251** (1960), 1258–1260.

If, in certain families of continuous curves in R_3, scalar product, zero element and norm are defined as in the note cited in the preceding review, and addition is defined in a natural way using any intrinsic parametrization, then the resulting spaces satisfy all of the axioms for Banach spaces, except possibly the cancellation law for addition and (consequently) the associative law for addition. The main example is the family of rectifiable curves; it is shown that, with the obvious choice of parameter, the resulting space is, in fact, not a Banach space.

C. W. Kohls (Rochester, N.Y.)

8313:

Geba, K.; Semadeni, Z. Spaces of continuous functions. V. On linear isotonical embedding of $C(\Omega_1)$ into $C(\Omega_2)$. Studia Math. **19** (1960), 303–320.

[For part IV see Bessaga and Pełczyński, Studia Math. **19** (1960), 53–62; MR **22** #3971.] Let Ω_1, Ω_2 be two compact Hausdorff spaces, and let $C(\Omega_1)$, $C(\Omega_2)$ denote the Banach lattices of all real continuous functions on Ω_1 and Ω_2 respectively. The authors study the relationship between a certain linear isotone embedding of $C(\Omega_1)$ into $C(\Omega_2)$ and the statement that Ω_1 is a continuous or homeomorphic image of a closed subset of Ω_2. Results concerning such relationship have been announced without proof in a previous paper by the same authors [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 187–189; MR **21** #4351]. The present paper gives proofs for these previously

announced theorems, which are now formulated with some modification and more detail. It is pointed out that the previously announced theorems 3 and 4 are incorrect. The hypothesis in these theorems that P is a non-negative projection of norm 1 from $C(\Omega_2)$ onto the embedded image of $C(\Omega_1)$ is not sufficient. To correct these theorems, the authors now require the additional hypothesis that P is a lattice homomorphism. *Ky Fan* (Detroit, Mich.)

8314:

Aubert, K. E. A representation theorem for function algebras with application to almost periodic functions. Math. Scand. **7** (1959), 202–210.

Let A denote a self-adjoint closed sub-algebra containing the unit of the Banach algebra of all complex-valued bounded functions on a space X. The author indicates two proofs of the isometric isomorphism of A to the Banach algebra $C(\mathfrak{M})$ of all complex-valued continuous functions on its maximal ideals space $\mathfrak{M}$ equipped with the Gelfand topology. This is actually a corollary to the Gelfand-Neumark representation theorem for Banach algebras. The author, however, proves it in the special case of function algebras by having as motivation its use in getting Weil's compactification of a topological group by its almost periodic functions. The proofs rely on Kadison's methods [Mem. Amer. Math. Soc. No. 7 (1951); MR **13**, 360]. It is pointed out that another proof is to be found in Dunford and Schwartz [#8302].

L. Nachbin (Waltham, Mass.)

8315:

Arens, Richard. The maximal ideals of certain function algebras. Pacific J. Math. **8** (1958), 641–648.

Let S denote the Riemann sphere (compactified complex plane), Z a compact subset of S, G an open subset of S satisfying $G \subset Z$. Denote by $H(G/Z)$ the algebra of functions continuous on Z and holomorphic on G, with norm equal to the maximum modulus on Z. Question: What is the maximal ideal space of $H(G/Z)$? The expected answer, namely Z itself, was known to be correct for the case $Z=S$ under certain conditions [Hoffmann and Singer, Royden: Symposium on Banach algebras and harmonic analysis, April, 1958; unpublished]. The present paper verifies the expected answer for general compact $Z \subset S$, provided only that $H(G/Z)$ is not reduced to constants.

The proof depends on refining a device originated by Hoffmann and Singer, which is used to establish the following approximation theorem: If $f \in H(G/Z)$, $z_0 \in Z$, $z_0 \neq \infty$, then there exists a sequence $(h_n) \subset H(G/Z)$ such that

$$\|f-f(z_0)-(z-z_0)h_n\| \to 0 \qquad (n \to \infty).$$

The construction of (h_n) is interesting. Each h_n is of the form

$$h(z) = \int (f(\zeta)-f(z))(\zeta-z)^{-1}\mu(d\zeta),$$

where μ is chosen from a special set B of Radon measures on the plane; if necessary, f is extended to the whole of S by putting it equal on $S-Z$ to a freely chosen constant λ such that $|\lambda| \leq \|f\|$. It is shown that such an h is holomorphic, not only at points of the co-support of μ (which is plain), but also (when $\mu \in B$) at all points of holomorphy of f. Continuity of h, even for general bounded Borel functions f, is another consequence of the special nature of μ, the proof here resembling arguments used in potential theory.

There are a few minor misprints: p. 642, line 13 up: for "J", read "I"; p. 646, line 7 up: for "$G(f)$", read "$F(f)$"; p. 647, line 17: for "m_n", read "E_n".

R. E. Edwards (London)

8316:

Goffman, K. [Hoffman, K.]; Zinger, I. M. [Singer, I. M.]. Some problems of Gel'fand. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 99–114. (Russian)

Let A be a closed, separating subalgebra with unit of $C(X)$, where X is compact Hausdorff. Let $S(A)$ denote the space of maximal ideals of A, $\Gamma(A)$ the Šilov boundary. A may be regarded as an algebra of functions on $S(A)$. Call A antisymmetric if it contains no non-constant real functions. Call A analytic if no function in A vanishes on an open subset of $S(A)$ unless it vanishes identically.

If G is an open subset of C^n, denote by $A(G)$ the algebra of all functions analytic in G, continuous in $\bar{G}$. If G is connected, $A(G)$ is clearly antisymmetric and analytic. If G is the unit disc in C^1, it is known that $A(G)$, considered as an algebra of functions on its Šilov boundary Γ, is a maximal subalgebra of $C(\Gamma)$ [J. Wermer, Proc. Amer. Math. Soc. **4** (1953), 866–869; MR **15**, 440; also Hoffman and Singer, Amer. J. Math. **79** (1957), 295–305; MR **19**, 46]. I. M. Gel'fand [same Uspehi **12** (1957), no. 1 (73), 249–251; MR **18**, 913] asked several questions designed to test how far $A(G)$ is typical among antisymmetric or analytic algebras. The authors construct antisymmetric algebras A with $S(A)$ homeomorphic to a three-dimensional cell, or a cell of any dimension ≥ 2, answering one of Gel'fand's questions. They construct an antisymmetric algebra A with $S(A)$ the unit ball B^4 in C^2, $\Gamma(A)$ the unit sphere S^3, such that A contains $A(B^4)$ as a proper subalgebra. This example shows that antisymmetric algebras need not be analytic, though analytic algebras are always antisymmetric, and that $A(B^4)$ is not a maximal subalgebra of $C(S^3)$.

Let X be a compact plane set, with positive measure and no interior. Let A be the algebra of all functions continuous on the Riemann sphere S and analytic in $S-X$. Wermer showed [Ann. of Math. (2) **62** (1955), 269–270; MR **17**, 255] that A contains enough functions to separate points of S. The authors show that if X has positive upper density at each point, then $S(A)=X$. More generally, if G is an open plane set, $X \subset G$, and $S(A(G))=\bar{G}$, then $S(A(G-X))=\bar{G}$. {Arens [#8315 above] has extended the argument to show that the upper density hypothesis and the hypothesis $S(A(G))=\bar{G}$ are unnecessary.} This result answers two more of Gel'fand's questions. First, it is possible for an antisymmetric algebra to have as maximal ideal space the 2-sphere. Second, $A(G)$ is never a maximal antisymmetric algebra with space of maximal ideals $\bar{G}$.

The authors leave unanswered two questions: (1) Is there an antisymmetric A with $S(A)$ homeomorphic to the unit interval I? (2) If A is antisymmetric, $S(A)$ a plane set, is A isomorphic to a closed subalgebra of $A(G)$ for some open plane set G? It is shown that if the answer to (1) is yes, then the answer to (2) is no.

{Hoffman has observed that (1) is equivalent to the question: are there any proper subalgebras A of $C(I)$ with $S(A)=I$? Rossi has shown [#8317 below] that, for any such

$A, \Gamma(A) = I$. Wermer has shown [Ann. of Math. (2) **68** (1958), 550–561; MR **20** #6536] that if A is generated by n sufficiently smooth functions, and $S(A) = I$, then $A = C(I)$.}

A. Browder (Providence, R.I.)

8317:

Rossi, Hugo. The local maximum modulus principle. Ann. of Math. (2) **72** (1960), 1–11.

Let X be a compact Hausdorff space. Let $\mathscr{C}(X)$ be the Banach algebra (sup norm) of continuous complex functions on X. Let A be any closed subalgebra of $\mathscr{C}(X)$ for which X "is" the space of maximal ideals. The Šilov boundary $\Gamma(A)$ is the smallest closed set in X on which each f in A attains its maximum modulus. The main theorem proved here is the following. If $x \in X - \Gamma(A)$ and U is a neighborhood of x, then for each f in A, $|f(x)|$ is not greater than the maximum of $|f|$ on the boundary of U. The (indirect) proof may be roughly described as follows. One presumes that U does not meet Γ and that $\max_U |f| > \max_{\bar{U}-U} |f|$. This situation is then transferred into a finitely generated subalgebra F of A, by use of a lemma of Calderón and the reviewer (and L. Waelbroeck, independently). Thus X can now be regarded as a polynomially-convex subset of $\mathbf{C}^n$, and is thus the intersection of open Stein manifolds, as Šilov first observed. A proposition to the effect that each component of a hull is also a hull is now applied to reduce the set in U on which $|f|$ is a maximum to a single point. Then f is deformed, unimportantly in U, but outside U in such a way that (the new) f has its maximum modulus only in U. This deformation is accomplished by solving a Cousin problem of the second kind presented by f and 1, in a Stein neighborhood of X. {The result, incidentally, can be extended to (the maximal ideal space of) any commutative Banach algebra with unit. However, the author chooses to rely on the following, which may not hold in general.} For a finitely-generated algebra (sup-normed, as above) if the set

$$\{|f| = \max_U |f|\} \cap U$$

is a single interior point x_0 of U, then $g(x_0) = 1$ for some g in A such that $|g(x)| < 1$ for all other $x \in X$. {A correction: after 5.1 (i) insert 'K is compact'.}

R. Arens (Los Angeles, Calif.)

8318:

Hoffman, K.; Singer, I. M. Maximal algebras of continuous functions. Acta Math. **103** (1960), 217–241.

Let X be a compact space, and A a closed linear subalgebra of the Banach algebra $\mathscr{C}(X, \mathbf{C})$ which lies in no proper subalgebra. Then A is called maximal. If $\mathbf{C}$ here is replaced by $\mathbf{R}$, the theory is quite simple, by virtue of the Stone-Weierstrass theorem, with these results: (i) a maximal algebra is either just the annihilator of some point x of X or it is the non-distinguisher of two points $x \neq y$; (ii) every closed subalgebra is the intersection of maximal algebras. One naturally wonders what happens in the complex case. The present paper practically indicates that no results of this sort are possible, by showing two things: (1) H^∞, the algebra of those measurable bounded functions on the unit circle for which $\int f(t) \exp(nit)dt = 0$, $n = 1, 2, \cdots$, is contained in no maximal subalgebra of L^∞ (the latter being regarded as $\mathscr{C}(X, \mathbf{C})$ where X is the space of maximal ideals of L^∞). (2) There exist maximal algebras demonstrably more bizarre than those described to date by Wermer, Rudin, Bishop and the authors. (Thus the assurance that a given subalgebra A is the intersection of maximal ones is of little use unless the maximal ones involved can be specified.) Nevertheless, the establishment of the maximality of any particular function algebra A contained in $\mathscr{C}(X, \mathbf{C})$ always results in approximation theorems. The paper contains a study of maximal algebras, relating maximality to "analyticity", "integrity", "antisymmetry" and "pervasiveness". The last means that for each proper closed subset Y of X, A generates a dense subset of $\mathscr{C}(Y, \mathbf{C})$. These theorems are of interest by themselves, but their main purpose is to help establish the remarkable facts (1), (2) stated above.

R. Arens (Los Angeles, Calif.)

8319:

Bauer, Heinz. Konservative Abbildungen lokal-kompakter Räume. Math. Ann. **138** (1959), 398–427.

Let φ be a continuous mapping of locally compact Hausdorff spaces, $X \rightarrow Y$; let $C(X)$, $C(Y)$ denote the algebras of continuous real functions with compact support in X, Y; and let $M(X)$, $M(Y)$ denote the positive Radon measures in X, Y. Then φ induces a mapping of $M(X)$ to $M(Y)$, also denoted by φ and defined by the equation $\int f d\varphi(\mu) = \int f(\varphi) \cdot d\mu$. The mapping φ is (here) called conservative if φ maps $M(X)$ onto $M(Y)$. Call φ intrinsic (eigentlich) if $\varphi^{-1}K$ is compact in X for any compact K in Y, and call φ locally intrinsic if the image of X in Y is a locally compact subspace and if φ is intrinsic as a mapping onto the image. Theorem: Every locally intrinsic φ is conservative. Theorem: Every φ is the restriction of an intrinsic mapping. However, not every φ is conservative. On the other hand, we have the theorem: if X is σ-compact (countable at ∞), every φ is conservative. Theorem: If φ has local cross-sections, φ is conservative. Some results on factoring of conservative transformations are given, and the paper concludes with a note concerning an application to Lebesgue surface measure.

F. B. Wright (New Orleans, La.)

8320:

Schwartz, J. T. Another proof of E. Hopf's ergodic lemma. Comm. Pure Appl. Math. **12** (1959), 399–401.

The discrete form of the lemma reads: Let a k-dimensional matrix $P = (p_{mn})$ satisfy $p_{mn} \geqq 0$ ($m, n = 1, 2, \cdots, k$) and $\sum_{m=1}^k p_{mn} \leqq 1$ ($n = 1, 2, \cdots, k$). Then, if $v = (v_1, \cdots, v_k)$ is a vector such that $v_m^* = \sup_{0 \leqq j < \infty} (v + Pv + \cdots + P^j v)_m \geqq 0$ for each $m = 1, 2, \cdots, k$, it follows that $\sum_{n=1}^k v_n \geqq 0$. As is expounded in Dunford and Schwartz [#8302, p. 684], the lemma is the basis of a systematic treatment of individual ergodic theorems concerning strongly measurable semi-groups of operators $\{T_t\}$ satisfying $\|T_t\|_{L_1(S)} \leqq 1$ and $\|T_s\|_{L_\infty(S)} \leqq 1$, where S is a positive measure space. The author devises a short proof which is much more transparent than the one given in the book referred to above.

K. Yosida (Tokyo)

8321:

Gårding, L.; Lions, J. L. Functional analysis. Nuovo Cimento (10) **14** (1959), supplemento, 9–66.

This paper is based on a series of lectures on the theory of distributions given by the authors. It covers the main points of Schwartz's theory as well as some special topics (see below), and includes a large number of exercises. The detailed contents are as follows. (1) Graphs, functions and groups. (2) Linear spaces. (3) Seminorms and topology.

(4) Complete spaces. (5) Continuous linear functions. (6) The technical theory of distributions. (7.1) Fourier transform. (7.2) Convolution and multiplication with Fourier transforms. (8) Lorentz invariant distributions. (9) Fourier transforms of Lorentz invariant distributions. Sections 8 and 9 contain a simplified version due to Roos and Gårding of the work by Methée [Comment. Math. Helv. **28** (1954), 225–269; C. R. Acad. Sci. Paris **240** (1955), 1179–1181; MR **16**, 255, 1101]. (10) Laplace transform. (11) Vectorial distributions. (12) The nuclear theorem of Schwartz. The exposition in section 12 is based on the method of Ehrenpreis [Proc. Amer. Math. Soc. **7** (1956), 713–718; MR **18**, 584].

J. Korevaar (Madison, Wis.)

8322:

Schwartz, Laurent. Théorie des distributions à valeurs vectorielles. II. Ann. Inst. Fourier. Grenoble **8** (1958), 1–209.

C'est le deuxième chapitre d'un mémoire dans lequel l'auteur étend aux distributions à valeurs vectorielles les résultats principales contenus dans sa *Théorie des distributions* [I et II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833]; le premier chapitre a paru dans les mêmes Ann. **7** (1957), 1–147 [MR **21** #6534].

L'objet essentiel est le problème (posé au § 1) de transporter des produits divers des distributions (produit scalaire, multiplication, convolution) sur le cas des distributions à valeurs vectorielles. Étant donné un produit $(S, T) \to S \cup T$ de $\mathscr{H} \times \mathscr{K}$ dans $\mathscr{L}$ (où $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$ sont 3 espaces des distributions) et une application bilinéaire θ de $E \times F$ dans G (où E, F, G sont 3 espaces vectorielles topologiques; on se ramène au cas de l'application bilinéaire canonique: $(\mathbf{e}, \mathbf{f}) \to \mathbf{e} \otimes \mathbf{f}$ de $E \times F$ dans $E \otimes F$ complété dans une topologie convenable), on cherche à construire un produit correspondant $(\mathbf{S}, \mathbf{T}) \to \mathbf{S} \cup_\theta \mathbf{T}$ de $\mathscr{H}(E) \times \mathscr{K}(F)$ dans $\mathscr{L}(G)$ ($\mathscr{H}(E) = \mathscr{H} \varepsilon E$, etc., défini dans le premier chapitre, étant l'espace des distributions à valeurs dans E correspondant à $\mathscr{H}$), vérifiant $(S\mathbf{e}) \cup_\theta (T\mathbf{f}) = (S \cup T)\theta(\mathbf{e}, \mathbf{f})$. La construction d'un tel produit n'est possible que dans certaines conditions assez restrictives (mais inévitables).

Le théorème général "de croisement" (§ 2, proposition 2) donnera des produits cherchés. Il étend l'application canonique $(L \otimes U) \times (M \otimes V) \to (L \otimes M) \otimes (U \otimes V)$ (où L, M, U, V sont 4 espaces localement convexes séparés) en donnant l'application $\Gamma_{\mu,\lambda}: (L \hat{\otimes}_\lambda U) \times (M \varepsilon V) \to (L \hat{\otimes}_\mu M) \varepsilon (U \hat{\otimes}_\lambda V)$; $L \hat{\otimes}_\lambda U$, etc., est le quasi-complété de $L \otimes U$ muni de la topologie λ; on suppose que λ, μ sont choisies parmis les 5 topologies (sur le produit tensoriel) $\varepsilon \leqq \pi \leqq \beta \leqq \gamma \leqq \iota$ (introduites au § 1; "$\leqq$" signifie "est moins fine que"), et que $\lambda \leqq \gamma$, $\mu \leqq \gamma$. Beaucoup de propriétés de l'application $\Gamma_{\mu,\lambda}$ sont établies. Le théorème de croisement entraîne la proposition 3 (§ 2) qui donne le produit $\cup_\pi$: $\mathscr{H}(E) \times \mathscr{K}(F) \to \mathscr{L}(E \hat{\otimes}_\pi F)$ (θ est ici l'application bilinéaire canonique de $E \times F$ dans $E \hat{\otimes}_\pi F$).

Le produit scalaire $\boldsymbol{\varphi} \cdot_\pi \mathbf{T}$ de $\mathscr{H}(E) \times \mathscr{H}'(F)$ dans $E \hat{\otimes}_\pi F$ est défini dans le § 3 (la proposition 4 donne des conditions d'existence; le dual $\mathscr{H}'$ de $\mathscr{H}$ étant un espace des distributions). Le produit scalaire est nul, lorsque les supports sont sans point commun. Il ne dépend pas de $\mathscr{H}$ et s'exprime par un intégrale (si $\boldsymbol{\varphi}$ et $\mathbf{T}$ sont des fonctions) dans certaines conditions supplémentaires.

Le § 4 est consacré à l'étude plus difficile d'un produit scalaire $\boldsymbol{\varphi} \cdot_\iota \mathbf{T}$ à valeurs dans $E \hat{\otimes}_\iota F$ (il ne s'obtient pas du théorème de croisement). Il y a des propositions analogues à celles du § 3, et les conditions, dans lesquelles tous les deux produits coïncident, sont établies. L'auteur donne trois exemples intéressants. (1) En prenant pour θ l'application: $(u, \mathbf{f}) \to u(\mathbf{f})$ de $E \times F$ dans G, où E est l'espace des applications linéaires continues de F dans G, on obtient un exemple utilisé par F. Bruhat [Bull. Soc. Math. France **84** (1956), 97–205; MR **18**, 907]. (2) Dans certaines conditions on trouve le dual de $\mathscr{H}(E)$. (3) On obtient le produit scalaire de $\overline{\mathscr{D}^{m+n+1}}(E) \times \mathscr{D}_c'^m(F)$ dans $E \hat{\otimes}_\iota F$ (le 2ième contre-exemple du § 8 montre qu'on ne peut pas remplacer $m+n+1$ par $m+n-1$); certains lemmes donnent une décomposition de δ en somme de dérivées des fonctions, et une caractérisation des opérateurs de convolution nucléaires.

Le produit multiplicatif de $\mathscr{H}(E) \times \mathscr{K}(F)$ dans $\mathscr{L}(E \hat{\otimes}_\pi F)$ (ainsi que celui à valeurs dans $\mathscr{L}(E \hat{\otimes}_\iota F)$ qui ne s'obtient pas du théorème de croisement) est considéré dans le § 4. Des conditions convenables assurent l'existence, l'associativité, l'inclusion des supports, la formule de dérivation, l'expression d'un produit scalaire par l'intégrale d'un produit multiplicatif ainsi que le fait que le produit des fonctions est une fonction. Les cas particuliers importants des espaces $\mathscr{H}$, $\mathscr{K}$ sont discutés séparément. Il y a deux exemples. (1) Multiplication des distributions semi-régulières (l'un en x, l'autre en y [cf. L. Schwartz, J. Math. Pures Appl. (9) **36** (1957), 109–127; MR **19**, 868]). (2) Sections-distributions des espaces fibrés à fibre vectorielle topologique.

Le produit tensoriel de deux distributions est traité brièvement au § 6.

Le § 7 est consacré au produit de convolution. Comme dans les cas précédents on en considère deux variantes: celui à valeurs dans $\mathscr{L}(E \hat{\otimes}_\pi F)$ et celui à valeurs dans $\mathscr{L}(E \otimes_\iota F)$. Dans des conditions convenables on établit l'existence, l'associativité, la commutativité, l'indépendance de $\mathscr{H}$ et $\mathscr{K}$, l'inclusion des supports, des formules avec un produit scalaire (et la régularisation). On considère aussi une autre définition de convolution à partir du produit tensoriel. On examine la possibilité d'exprimer la convolution par une intégrale partielle (le cas des fonctions est traité séparément). On donne enfin des relations entre la convolution et la multiplication par les transformations de Fourier et de Laplace.

Le dernier § 8 contient trois contre-exemples: (1) d'une fonction indéfiniment dérivable et d'une distribution, dont la convolution n'est pas une fonction, (2) d'une fonction k-fois continuement différentiable (sur R^n, $k \leq n-2$) et d'une mesure, dont la convolution n'est pas une fonction, (3) de deux mesures dont la convolution n'est pas une mesure; évidemment toutes ces fonctions, mesures et distributions sont à valeurs vectorielles.

S. Łojasiewicz (Kraków)

8323:

Łojasiewicz, S.; Zieleżny, Z. Une remarque sur la multiplication des distributions de la classe H_∞. Colloq. Math. **7** (1959), 57–60.

Let H_∞ be the class of distributions T on $(-\infty, \infty)$ such that T and all its derivatives are of the form $f+U$, where f is a locally summable function and U is a distribution whose support is an isolated set. A. César de Freitas [Univ.

Lisboa. Revista Fac. Ci. A (2) **5** (1955/56), 135–146; MR **17**, 1190] introduced an operation of multiplication, associative but not commutative, that is applicable to this class. The authors show that if $T \cdot S$ is a bilinear associative operation, defined on $H_\infty \times H_\infty$, of local character (i.e., if $T = T_1$ and $S = S_1$ in an interval Δ, then $T \cdot S = T_1 \cdot S_1$ in Δ), and satisfying $(T \cdot S)' = T' \cdot S + T \cdot S'$, and if the operation reduces to ordinary multiplication when T and S are functions, then it is essentially multiplication in the sense defined by César de Freitas. In particular, it satisfies

$$\delta_a^{(i)} \cdot \delta_a^{(j)} = 0 \quad (i, j = 0, 1, 2, \cdots),$$

and either

$$\delta_a \cdot f = f(a-0)\delta_a, \quad f \cdot \delta_a = f(a+0)\delta_a,$$

or

$$\delta_a \cdot f = f(a+0)\delta_a, \quad f \cdot \delta_a = f(a-0)\delta_a.$$

F. Smithies (Cambridge, England)

8324:

Guerreiro, J. Santos. La multiplication des distributions comme application linéaire continue. Portugal. Math. **18** (1959), 55–67.

The author investigates the possibility of defining the multiplication of an arbitrary distribution by a function, and proves the following results. For the class C_π of distributions on R^n, every continuous linear mapping of C_π into itself that commutes with multiplication by the coordinate functions reduces to the operation of multiplication by a function $\varphi \in C^\infty$; the same is true for every continuous linear mapping whose restriction to the class of continuous functions reduces to the operation of multiplying by a continuous function. In other words, one can only get a sensible definition of multiplication of an arbitrary distribution by a function when the function is of class C^∞. For the class of distributions on the closure Δ of a bounded open subset of R^n, the corresponding class of functions consists of those functions φ that are indefinitely differentiable in the interior of Δ and such that φ and all its derivatives are of slow growth towards the frontier of Δ. *F. Smithies* (Cambridge, England)

8325:

Vasilach, Serge. Calcul opérationnel algébrique des distributions à support dans R_+^n, $n \geq 1$. I. Rev. Math. Pures Appl. **4** (1959), 185–219.

It follows from the theorem of Titchmarsh-Crum-Dufresnoy that the ring $\mathscr{C}_+$ of continuous functions with support in the half-line $[0, \infty)$ under the operations of addition and convolution has no zero divisors. The elements of the field of quotients of $\mathscr{C}_+$ are the operators of Mikusiński [Studia Math. **11** (1949), 41–70; MR **12**, 189]. Subsequently L. Schwartz proved [*Théorie des distributions, Tome II*, Actualités Sci. Ind. no. 1122, Hermann, Paris, 1951; MR **12**, 833; cf. p. 29] that the ring $\mathscr{D}_+'$ of distributions with support in $[0, \infty)$ is also a domain of integrity. The author considers the field of quotients $\mathbf{K}$ of $\mathscr{D}_+'$ and in particular the Heaviside function Y and its inverse (in $\mathbf{K}$) $p = \delta'$, where the Dirac measure δ is the unit element of $\mathbf{K}$. He proves some identities concerning algebraic expressions involving p and applies them to the solution of ordinary differential equations with constant coefficients and of integral equations of Volterra type.

J. Horváth (College Park, Md.)

8326:

Methée, Pierre-Denis. Systèmes différentiels du type de Fuchs en théorie des distributions. Comment. Math. Helv. **33** (1959), 38–46.

L'auteur considère l'équation $x dT/dx + AT = 0$, où T est une distribution vectorielle $\{T_1, \cdots, T_n\}$ et A une matrice indéfiniment dérivable et analytique au voisinage de l'origine. Il démontre que la solution générale de cette équation dépend de $2n$ constantes arbitraires.

G. Marinescu (Bucharest)

8327:

Pál, L. G. Über die kanonische Darstellung der linearen Operationen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 73–84.

L'A. démontre le théorème suivant: Une fonctionnelle linéaire A définie sur l'espace $C[a, b]$ des fonctions numériques continues sur le segment $[a, b]$ peut s'écrire sous la forme $A(f) = \int f(x)\beta(x)dx$ pour $f \in C[a, b]$ si et seulement si $A(f_n) \to 0$ pour toute suite (f_n) de fonctions de $C[a, b]$ uniformément bornée et convergente vers 0 presque partout. De ce théorème on déduit, en particulier, le théorème de représentation intégrale des fonctionnelles linéaires continues sur $L^p[a, b]$, $1 \leq p < \infty$. L'A. énonce aussi le théorème suivant: Soit $N[a, b]$ un espace linéaire de fonctions numériques mesurable Lebesgue sur $[a, b]$ tel que $\sup(f, g) \in N[a, b]$ si $f, g \in N[a, b]$. Une fonctionnelle linéaire A sur $N[a, b]$ admet une représentation intégrale de la forme précédente si et seulement si $A(f_n) \to 0$ pour toute suite (f_n) de fonctions de $N[a, b]$ convergente vers 0 presque partout et majorée par une fonction $f \in N[a, b]$, $|f_n(x)| \leq f(x)$. *N. Dinculeanu* (Bucharest)

8328:

Ghika, Al. Sous-espaces vectoriels orthogonaux dans les produits d'espaces hilbertiens. Com. Acad. R. P. Romîne **9** (1959), 763–766. (Romanian. Russian and French summaries)

Using the fact that a direct sum Σ of Hilbert spaces can be embedded in the direct product Π of the same spaces, the author notes that the annihilating manifold $M^\perp$ of a closed linear subspace M of either space satisfies $M + M^\perp$ is dense in Π. *M. M. Day* (Urbana, Ill.)

8329:

Ghika, Al. Le transposé d'un opérateur défini d'une manière dense dans un espace localement convexe. Com. Acad. R. P. Romîne **9** (1959), 985–988. (Romanian. Russian and French summaries)

The author gives a proof for the theorem that if A is a closed, densely defined linear operator in a locally convex space $\mathscr{F}$, then the adjoint operator A' is the same kind of operator in the dual space $\mathscr{F}'$ when the $\mathscr{F}[\mathscr{F}']$ topologies are used in $F'[F]$. For reflexive spaces the result holds also for the strong topologies. *M. M. Day* (Urbana, Ill.)

8330:

Ghika, Al. Sommes projectives d'espaces vectoriels topologiques. Com. Acad. R. P. Romîne **9** (1959), 1109–1112. (Romanian. Russian and French summaries)

The author observes that the space $L^2K(R)$ of functions of summable square on every bounded set is a topological

direct product of the Hilbert spaces $L^2(k, k+1)$, where k runs over the integers. *M. M. Day* (Urbana, Ill.)

8331a:

Ghika, Al. Transformations de certains espaces vectoriels métrisables en espaces de Hilbert. Com. Acad. R. P. Romîne **9** (1959), 1231–1236. (Romanian. Russian and French summaries)

8331b:

Ghika, Al. Transformations d'une classe d'espaces de Banach en espaces hilbertiens. Com. Acad. R. P. Romîne **10** (1960), 5–9. (Romanian. Russian and French summaries)

The author begins with the observation that the usual power homeomorphism between l^p and l^2 can be made into a vector space isomorphism by using the homeomorphism $\varphi(t)=t|t|^{p/2-1}$ to define a new addition and multiplication in the real (or complex) field. The second note carries this out in detail, but first states that any homeomorphism of a normed space E onto a Hilbert space determines a homeomorphism of the scalar field which can then be used as in the special case of the l^p spaces. Unfortunately, this assertion has hypotheses which are satisfied by homeomorphisms which do not carry rays through the origin into rays through the origin, and by other homeomorphisms which do take rays into rays but treat the scalar field differently on different rays. *M. M. Day* (Urbana, Ill.)

8332:

Čihák, Pavel. On the relative compactness of subsets of the space $L_p(T)$. Czechoslovak Math. J. **9 (84)** (1959), 334–338. (Russian. English summary)

Let T be a measurable set in E_n, and regard $L_p(T)$ ($p\geqq 1$) as a space of functions defined on all of E_n with support in T. For $r>0$, let S_r be the ball in E_n with center at the origin and radius r, and define

$$x_r(t)=\frac{1}{m(S_r)}\int_{S_r} x(t+s)\,dm(s)\quad (x\in L_p(T)).$$

The author proves that a set $X\subset L_p(T)$ is relatively compact if and only if (1) $\lim_{r\to 0+}\|x-x_r\|=0$ uniformly for $x\in X$, and (2) for each infinite set $Y\subset X$, there exist an infinite subset $Y_1\subset Y$ and a $w\in L_p(T)$ such that $|x|\leqq w$ for all $x\in Y_1$. [Cf. Dunford and Schwartz, #8302; pp. 297–301, 388.] *M. Jerison* (Lafayette, Ind.)

8333:

Dinculeanu, N. Espaces d'Orlicz de champs de vecteurs. IV. Opérations linéaires. Studia Math. **19** (1960), 321–331.

[For part III, see Studia Math. **17** (1958), 285–293; MR **21** #2182.] Recently the author has given a theorem [see N. Dinculeanu, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **22** (1957), 239–275; MR **19**, 1066] about the integral representation of a certain linear transformation of an Orlicz vector field into a Banach space. In this paper this theorem is generalized to the following form: condition (1) is dropped and then necessary and sufficient conditions are given for the validity of the integral representation. For details the reader is referred to the paper. *W. A. J. Luxemburg* (Pasadena, Calif.)

8334:

Wigner, Eugene P. Normal form of antiunitary operators. J. Mathematical Phys. **1** (1960), 409–413.

An operator, A, everywhere defined on a Hilbert space is anti-unitary if it is additive $(A(\Phi_1+\Phi_2)=A\Phi_1+A\Phi_2)$, has an inverse, and satisfies $(A\Phi, A\Psi)=\overline{(\Phi,\Psi)}$. The author shows that in the finite-dimensional case (extension of the results to the infinite-dimensional case is briefly indicated), one can characterize an anti-unitary operator by using the proper vectors of its square, A^2, which is unitary and real. The proper values of A^2 are complex numbers of modulus 1. The proper vectors belonging to ω are denoted by $v_{\omega k}(k=1,2,\cdots)$. Then, with a suitable choice of phase, $Av_{\omega k}=(\omega^{1/2})^*v_{\omega^* k}$. These sets of proper vectors are not uniquely determined. Those belonging to the proper value -1 may be replaced by linear combinations using a symplectic matrix. For complex ω one may use an arbitrary unitary transformation provided the vectors belonging to the complex conjugate are transformed with the complex conjugate unitary transformation. The vectors belonging to $\omega=1$ may be transformed with an arbitrary real orthogonal transformation. Using this information the author counts the number of parameters in a general anti-unitary transformation. *A. S. Wightman* (Princeton, N.J.)

8335:

Inoue, Sakuji. Normal operators in Hilbert spaces and their applications. Mem. Fac. Ed. Kumamoto Univ. **7**, suppl. 2, 1–36 (1959).

This is a neatly written, essentially expository paper on the elementary theory of a normal compact operator in Hilbert space; the results obtained are either well known or can easily be derived from the spectral decomposition of such operators. Also considered are normal operators whose spectrum has but a finite or countable number of accumulation points, and applications are made to linear integral equations of the second kind. *H. H. Schaefer* (Ann Arbor, Mich.)

8336:

Inoue, Sakuji. Corrections to the paper "Simplification of the canonical spectral representation of a normal operator in Hilbert space and its applications". Mem. Fac. Ed. Kumamoto Univ. **7**, suppl. 2, 37–38 (1959).

Additional assumptions have to be made in lemmas 4.1, 4.2 and, accordingly, in §§ 5, 6, 7 of the paper mentioned in the title [Inoue, same Mem. **3** (1955), suppl. 1; MR **18**, 661]. *H. H. Schaefer* (Ann Arbor, Mich.)

8337:

Inoue, Sakuji. An application of a compact normal operator in Hilbert spaces to the theory of functions. Proc. Japan Acad. **36** (1960), 128–132.

Unfortunately, due to a grave and irreparable error (p. 129, lines 4–5), none of the results of this paper are correct. The use of normal operators is not relevant to the questions involved; a counter-example to theorem 1 is furnished by the function $z\to\exp z^{-1}+\sum_1^\infty 2^{-n}/(\lambda-n^{-1})$. *H. H. Schaefer* (Ann Arbor, Mich.)

8338:

Sz.-Nagy, Béla. ★Extensions of linear transformations in Hilbert space which extend beyond this space (Appendix

to **Frigyes Riesz and Béla Sz.-Nagy, "Functional analysis")**. Translated from the French by Leo F. Boron. Frederick Ungar Publishing Co., New York, 1960. 37 pp. $2.50.

Reviews have already appeared of the French [Akad. Kiadó, Budapest, 1955; MR **16**, 837] and German [Schr. Forschungsinst. Math. **1** (1957), 289–302; MR **19**, 296] versions.

8339:

Bishop, Errett. A duality theorem for an arbitrary operator. Pacific J. Math. **9** (1959), 379–397.

Since results in spectral theory concern themselves with relations involving T and its adjoint T^*, the author speaks of duality theory rather than of spectral theory. He presents four types of duality theory which seem to embrace the known types of structure theory. All spaces considered are reflexive, although it is stated that the results may be modified so as to present a theory valid in any space. Throughout, T is a bounded operator on a complex reflexive space B, F is a closed subset of the complex plane, $-F$ is its complement, G is an open set, $\bar{G}$ is its closure, $f(z)$ is an analytic function with values in B.

Definitions: The 'strong spectral manifold' $M(F, T)$ is the closure of the set of $x \in B$ such that there exists f defined on $-F$ for which $(T-zI)f(z)=x$ for all z in $-F$. The 'weak spectral manifold' $N(F, T)$ is the set of all $x \in B$ such that for $\varepsilon>0$ there exists f defined on $-F$ such that $\|(T-zI)f(z)-x\|<\varepsilon$ for all z in $-F$. T 'admits a duality theory of type 1' if $M(F_1, T)^\perp \supset M(F_2, T^*)$ for compact disjoint F_1, F_2 and if $M(\bar{G}_1, T)^\perp \subset M(\bar{G}_2, T^*)$ whenever G_1 and G_2 cover the complex plane. T 'admits a duality theory of type 2' if $M(\bar{G}_1, T), \cdots, M(\bar{G}_n, T)$ span B whenever $G_1, \cdots, G_n$ cover the complex plane. T 'admits a duality theory of type 3' provided that for any $G_1, \cdots, G_n$ which cover the complex plane, there exist closed linear manifolds $M_1, \cdots, M_n$ reducing T and spanning B for which $\sigma(T/M_i)\subset \bar{G}_i$ – here $\sigma(T)$ means the spectrum of T. For example, the uniformly bounded operators T, $\|T^n\|<k$, treated by the reviewer [Trans. Amer. Math. Soc. **49** (1941), 18–40; MR **2**, 224] fall into this category. T 'admits a duality theory of type 4' provided that for arbitrary disjoint compact F_1, F_2, and G_1, G_2 which cover the complex plane, $M(F_1, T)^\perp \supset N(F_2, T^*)$, $N(F_1, T)^\perp \supset M(F_2, T^*)$, $M(\bar{G}_1, T)^\perp \subset N(\bar{G}_2, T^*)$, and $N(\bar{G}_1, T)^\perp \subset M(\bar{G}_2, T^*)$. T 'satisfies condition β' if for every open U and every sequence $\{f_n\}$ from U to B such that $(T-zI)f_n(z)\to 0$ uniformly on U, $\{f_n\}$ is uniformly bounded on compact subsets of U.

Theorem: Every T admits a duality theory of type 4. Definition: T 'satisfies condition α' if $N(F_1, T)\subset M(F_2, T)$ in case $F_1 \subset$ interior F_2. Theorem: If T satisfies α, it admits a duality theory of type 1. Theorem: If T^* satisfies β then T admits a duality theory of type 2. If T and T^* satisfy β then T admits a duality theory of type 3. There are other results not reported here.

E. R. Lorch (New York)

8340:

Foiaş, Ciprian. Certaines applications des ensembles spectraux. I. Mesure harmonique-spectrale. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 365–401. (Romanian. Russian and French summaries)

D'après J. von Neumann, un ensemble fermé S dans le plan des nombres complexes s'appelle ensemble spectral pour l'opérateur (linéaire) borné T de l'espace de Hilbert $\mathfrak{H}$ si l'on a $\|u(T)\|\leq \sup_{z\in S}|u(z)|$ pour toute fonction $u(z)$ holomorphe dans un ensemble ouvert contenant S. En généralisant ses recherches précédentes où il s'agissait seulement des ensembles S limités par une courbe jordanienne [Bull. Soc. Math. France **85** (1957), 263–282; MR **20** #7218], l'auteur considère dans le présent article un ensemble spectral S quelconque qui est borné et ne sépare pas le plan. On étend la correspondance $u(z)\to u(T)$ d'abord aux fonctions $u(z)$ qui sont harmoniques dans un ensemble ouvert contenant S (en prenant les parties réelles des fonctions holomorphes et des opérateurs correspondants), puis aux fonctions qui sont des limites uniformes, sur S, de telles fonctions ("fonctions S-harmoniques"). En vertu d'un théorème de Walsh [J. Reine Angew. Math. **159** (1928), 197–209] toute fonction continue $\varphi(\lambda)$ sur Fr S (frontière de S) peut être continuée univoquement à une fonction S-harmonique sur S. On a donc une correspondance linéaire $\varphi(\lambda)\to\varphi(T)$ telle que $\|\varphi(T)\|\leq \sup_{\lambda\in \mathrm{Fr}\,S}|\varphi(\lambda)|$. Cette correspondance se prolonge aux fonctions caractéristiques $\omega(\lambda;\beta)$ des sous-ensembles boréliens β de Fr S; les opérateurs correspondants autoadjoints $\omega(T;\beta)$ constituent la "mesure harmonique-spectrale" par rapport à T et S. On étude les propriétés de $\omega(T;\beta)$, liées au calcul avec des fonctions harmoniques (invariance par rapport à des représentations conformes, "principe de la mesure harmonique" de R. Nevanlinna, etc.). On gagne entre autres des informations importántes sur la classe $B_T(S)$ des ensembles boréliens $\beta\subset \mathrm{Fr}\,S$ pour lesquels $\omega(T;\beta)$ est une projection. En appliquant un théorème de Neumark on obtient qu'il existe, dans un espace de Hilbert $\mathfrak{K}\supseteqq\mathfrak{H}$, une mesure spectrale ordinaire $E(\beta)$ ($\beta\subset \mathrm{Fr}\,S$, borélien) telle que

$$(\omega(T;\beta)x, y) = (E(\beta)x, y) \quad (x, y\in\mathfrak{H}).$$

On étudie les propriétés de l'opérateur normal $N=\int_{\mathrm{Fr}\,S}\lambda dE(\lambda)$ ("dilatation normale" de l'opérateur T par rapport à S) et on étend ainsi certains résultats du travail de B. Sz.-Nagy et C. Foiaş, Acta Sci. Math. Szeged **19** (1958), 26–45 [MR **21** #2188]. *B. Sz.-Nagy* (Szeged)

8341:

Senčihina, I. N. Eigenfunction expansions of a difference operator with operator coefficients. Ukrain. Mat. Ž. **11** (1959), 183–191. (Russian. English summary)

In a number of papers Yu. M. Berezanskiĭ [Dokl. Akad. Nauk SSSR **93** (1953), 5–8; **97** (1954), 573–576; Trudy Moskov. Mat. Obšč. **5** (1956), 203–268; MR **16**, 713; **19**, 288] studied certain difference operators, among them (*) $L[u_j]\equiv a_{j-1}u_{j-1}+b_ju_j+a_ju_{j+1}=\lambda u_j$, where u_j is a vector in l_2 and a_j, b_j are certain infinite matrices. The present work extends the study of (*), where now $\{u_j\}$ is a sequence of elements in a Hilbert space and a_j, b_j are bounded self-adjoint operators in this space. Let H be a separable Hilbert space with scalar product (u, v); let $l_2(H)$ be the Hilbert space of all infinite sequences $u=\{u_j\}$, with $u_j\in H$ and $\sum\|u_j\|^2<\infty$, and with scalar product $\langle u, v\rangle = \sum(u_j, v_j)$. Finally, let $I_2(H)$ be the set of all infinite sequences $U=\{U_j\}$ of bounded operators in H such that series $\sum_j U_j{}^*U_j$ converges weakly. $I_2(H)$ is a Banach space with generalized scalar product $\{U, V\}=\sum_j U_j{}^*V_j$ and convergence defined by $U^{(n)}\to 0$ if $(\{U^{(n)}, U^{(n)}\}x, x)\to 0$ for all $x\in H$.

In this setting the author considers equation (*) above, with which is associated the difference operator equation $a_{j-1}U_{j-1}+b_jU_j+a_jU_{j+1}=\lambda U_j$, for sequences $U=\{U_n\}$ of bounded operators. Let A be an operator in $l_2(H)$, whose domain $D(A)$ is the set of all finite sequences from $l_2(H)$, defined by $(Au)_j=L[u]_j$ $(u\in D(A))$. If $\tilde{A}$ is any self-adjoint extension of A in $l_2(H)$, it is shown that there corresponds an operator function $T(\lambda)$ with respect to which every pair F, G of finite sequences of $I_2(H)$ satisfy the Parseval equation $\{F, G\}=\int_{-\infty}^{\infty} F^*(\lambda)d_\lambda T(\lambda)G(\lambda)$.

I. M. Sheffer (University Park, Pa.)

8342:

Neubauer, Gerhard. Zu einem Satz von N. Dunford. Arch. Math. **11** (1960), 366–367.

Let T be a bounded linear spectral operator (in the sense of Dunford) in a complex Banach space X, and let its quasinilpotent part be N. This paper gives the following necessary and sufficient condition that $N^m=0$: There exists a positive constant M (depending only on T) such that if $\sigma_1, \cdots, \sigma_n$ is any finite collection of pairwise disjoint Borel subsets of $\sigma(T)$ and $\xi_1, \cdots, \xi_n$ are any complex numbers such that $|\xi_j|\leq\|T\|+1$ and ξ_j is not in the closure of σ_j, then

$$\Big\|\sum_{j=1}^{n} R_{\xi_j}(T\,|\,E(\sigma_j)X)E(\sigma_j)\Big\| \leq M\delta^{-m},$$

where δ is the least of the distances dist $(\xi_j, \bar{\sigma}_j)$. Here $E(\cdot)$ is the resolution of the identity for T and, for λ not in $\bar{\sigma}$, $R_\lambda(T\,|\,E(\sigma)X)$ is the resolvent of the restriction of T to $E(\sigma)X$.

This theorem is related to a theorem of Dunford dealing with the same question when X is a Hilbert space [N. Dunford, Pacific J. Math. **4** (1954), 321–354; MR **16**, 142]. In that case, as Dunford proved, it is necessary and sufficient to have the condition here stated merely with $n=1$. It is known [C. A. McCarthy, ibid. **9** (1959), 1223–1231; MR **21** #7442] that Dunford's form of the condition is not sufficient when X is a Banach space. It implies $N^{m+2}=0$, but no more than this, in general.

A. E. Taylor (Los Angeles, Calif.)

8343:

Gel'man, A. E. Small parameter method for operator equations. Dokl. Akad. Nauk SSSR **123** (1958), 782–784. (Russian)

Let Y be a Banach space and let Y_λ be the linear space of all formal power series $\{y(\lambda)=\sum y_k\lambda^k\}$ with coefficients in Y. Suppose that Ω_λ maps Y_λ into Y_λ so that

$$\Omega_\lambda(y(\lambda)) = \Omega_0(0)+\sum_{k=1}^{\infty}\lambda^k\omega_k(y_0, \cdots, y_{k-1})$$

and that there is a double power series $\bar{\Omega}$ with positive coefficients and positive radius of convergence R such that when $y(\lambda)$ in Y_λ is termwise dominated by a positive series $x(\lambda)$ with $x(0)<R$ then $\Omega_\lambda(y(\lambda))$ is termwise dominated by $\bar{\Omega}(\lambda, x(\lambda))$. Suppose also that $\|\Omega_0(0)\|<R$. Then the equation $y=\Omega_\lambda(y)$ has a unique solution in Y_λ and the solution converges in a circle whose radius can be estimated.

R. G. Bartle (Urbana, Ill.)

8344:

Foguel, S. R. Finite dimensional perturbations in Banach spaces. Amer. J. Math. **82** (1960), 260–270.

Let T be a bounded linear operator on a Banach space X, and let S be a linear one-dimensional operator, i.e., $Sx=x_0^*(x)x_0$. The author studies the spectral properties of the operator $T+S$. His main results are the following: (1) If $\mu\in\sigma(T)$ and $x_0^*((\mu I-T)^{-1}x_0)\neq 1$ [resp. $=1$], then $\mu\notin\sigma(T+S)$ [resp. μ is an eigenvalue of $T+S$ with a single eigenvector]. (2) If μ is an isolated point of $\sigma(T)$, and F, M are the projection and quasinilpotent associated with T at μ, then (i) if $M=0$ and $x_0^*(Fx_0)\neq 0$ there is no nilpotent associated with $T+S$ at μ, and $\mu\notin\sigma(T+S)$ if and only if FX is a one-dimensional space; (ii) if $M^n=0$ and if E, N are the projection and quasinilpotent associated with $T+S$ at μ, N is nilpotent, and $E-F$, $N-M$ are finite-dimensional operators. An application to ordinary differential operators is also given.

C. Foiaş (Bucharest)

8345:

Kuroda, Shige Toshi. Perturbation of continuous spectra by unbounded operators. I. J. Math. Soc. Japan **11** (1959), 246–262.

For stability of the absolutely continuous part of the spectrum of a self-adjoint operator H_0 under the addition of a self-adjoint perturbation V, the best possible condition on V irrespective of H_0 is in a sense [cf. Kuroda, Proc. Japan Acad. **34** (1958), 11–15; MR **21** #1537] that of Kato [ibid. **33** (1957), 260–264; MR **19**, 1068], that V have an absolutely convergent trace. The present author extends Kato's work by obtaining an H_0-dependent condition applicable to situations in quantum mechanics outside the scope of Kato's result. Specifically he shows that if H_0 is self-adjoint and V is symmetric in a Hilbert space, and if (i) the domain of H_0 is contained in the domain of V; (ii) $\|Vu\|\leq a\|H_0u\|+b\|u\|$ for all vectors u in the domain of H_0, where a and b are constants such that $0\leq a<1$ and $0\leq b$, then $H=H_0+V$ is self-adjoint; and if (iii) $|V|^{(1/2)}(H_0-a)^{-1}$ is of absolutely convergent trace for some number a in the resolvent set of H_0, then the (so-called wave operators) $\lim_{t\to\pm\infty}\exp(itH)\exp(-itH_0)P_0$ exist, where P_0 is the projection whose range is the absolutely continuous subspace with respect to H_0. He also establishes the continuous dependence of these limits on V, in a certain sense.

The proof uses methods and results due to Kato as well as a method employed by Hille in his work on semi-groups. The present results, however, unlike the original ones of Kato, apply to the Schrödinger equation for a particle in a space of dimension not more than three, acted on by a potential which is both integrable and square-integrable.

I. E. Segal (Cambridge, Mass.)

8346:

Raimi, Ralph A. On Banach's generalized limits. Duke Math. J. **26** (1959), 17–28.

Let S be a set of points. Let E be a Banach space of bounded real-valued functions on S with the norm $\|f\|=\sup_{x\in S}|f(x)|$, and let E be translation-invariant with respect to a set K of linear operations of E into itself, i.e., if $f\in E$ then Tf is in E again, for any f in E and T in K. By generalized Banach limit the author means a positive linear functional ϕ' over E such that $(\phi', Tf)=(\phi', f)$, for any f in E and T in K.

The principal result of the paper concerns the case where $K=G$ is an Abelian semigroup with unit and S is the set of all points of G. The author gives a characteriza-

tion of the set of all generalized Banach limits and the answer to the question: what are their extreme values on a given function? More exactly: Let, for $g \in G$, T_g denote the linear operation such that $T_g f = f(g)$, for every f in E. Let L' be the set of all generalized Banach limits over E and let I be the set of all linear mappings of E into itself of the form $T_\alpha = \sum_{i=1}^n a_i T_{g_i}$, where $\sum_{i=1}^n a_i = 1$, $a_i > 0$ and $g_i \in G$ ($i = 1, 2, \cdots, n$; $n = 1, 2, \cdots$). Order I by the relation $T_\alpha > T_\beta$ if there exists T_γ such that $T_\alpha = T_\gamma T_\beta$. Let z be a point in S such that $G \cdot z = G = S$, and let z' be its corresponding linear functional, i.e., $(z', f) = f(z)$ for any $f \in E$. Further, assume that E is a separating set of functions on S and that the unit function $u(x) = 1$ is in E. Then

(a) $$L' = \bigcap_{T_\beta \in 3} \overline{\bigcup_{T_\alpha > T_\beta} T_\alpha' z'},$$

where the bar denotes the closure in weak topology of functionals and T_α' denotes the conjugate operation to T_α; (b) for any $f \in E$, $\sup_{\phi' \in L'} (\phi', f) = \lim \sup_\alpha (z', T_\alpha f)$; (c) for any $f \in E$, $\inf_{\phi' \in L'} (\phi', f) = \lim \inf_\alpha (z', T_\alpha f)$.

The case where $K = G = S$ happens to be ergodic in the sense of M. M. Day [Trans. Amer. Math. Soc. **69** (1950), 276–291; MR **13**, 357]. The author considers also a non-ergodic example in which K is the set of operators U_ρ: $f(x) \to \rho^{-1} \int_0^\rho f(x+t)\,dt$, $\rho > 0$ and S is the closed interval $[0, \infty)$. *A. Pelczynski* (Warsaw)

8347:

Kato, Tosio. Remarks on pseudo-resolvents and infinitesimal generators of semi-groups. Proc. Japan Acad. **35** (1959), 467–468.

In this note it is shown that if X is a locally sequentially weakly compact Banach space and if A is an operator on X whose resolvent satisfies the requirements of an infinitesimal generator of a semi-group of bounded operators, then A is necessarily densely defined. This fact is obtained as a corollary to a general theorem on pseudo-resolvents—i.e., families of bounded operators satisfying the resolvent equation. This latter theorem states that under suitable conditions the null space N and range R of a pseudo-resolvent do not intersect and that if X is locally sequentially weakly compact then X is the direct sum of N and the closure of R. *G. Hufford* (Seattle, Wash.)

8348:

Butzer, P. L.; Tillmann, H. G. Approximation theorems for semi-groups of bounded linear transformations. Math. Ann. **140** (1960), 256–262.

Let X be a Banach space, $\{T(t)\}$ a one-parameter semi-group of operators on X having infinitesimal generator A. The authors study the degree of approximation of $T(t)f$, for f in the domain of A^{p-1}, by the "Taylor polynomial" $\sum_{k=0}^{p-1} (t^k/k!) A^k f$. They show, subject to conditions on $T(t)$ for t near 0, that the degree is $o(t^p)$ if and only if $A^p f = 0$; and if X is reflexive, the degree is $O(t^p)$ if and only if f is in the domain of A^p. *K. deLeeuw* (Princeton, N.J.)

8349:

Tillmann, Heinz Günther. Approximationssätze für Halbgruppen von Operatoren in topologischen Vektorräumen. Arch. Math. **11** (1960), 194–199.

The results of the above paper [#8348] are extended to one-parameter semi-groups of operators on topological linear spaces. *K. deLeeuw* (Princeton, N.J.)

8350:

Knyazev, P. N. On a condition of preservation by monotone functions of the order relation for operators. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 141–146. (Russian)

Let A and B be bounded non-negative self-adjoint operators, $A(t)$ and $B(t)$ their corresponding resolutions of the identity. In order that $f(A) \geqq f(B)$ for every continuous monotone non-decreasing function f, it is necessary and sufficient that $A(t) \leqq B(t)$ for all t. The proof is an application of integration by parts.

Let A and B be as above and in addition completely continuous. Then $A \geqq B \geqq 0$ implies $A(t) \leqq B(t)$ if and only if there exists a sequence $\{A_i\}$ of operators satisfying the conditions: (1) $A \geqq A_1 \geqq A_2 \geqq \cdots \geqq B$; (2) A_{i-1} permutes with A_i; (3) ($\lim A_i$) permutes with B. If the complete continuity condition is removed then these three conditions are sufficient for the validity of the theorem. In the latter case some necessary conditions are also given. *A. Devinatz* (Princeton, N.J.)

8351:

Šaškin, Yu. A. Convergence of positive linear operators in the space of continuous functions. Dokl. Akad. Nauk SSSR **131** (1960), 525–527 (Russian); translated as Soviet Math. Dokl. **1**, 303–305.

Let Q be a compact metric space, and let $C(Q)$ and $B(Q)$ denote, respectively, the continuous and bounded real functions on Q. Let $\{L_n\}$ be a fixed sequence of positive linear operators from $C(Q)$ into $B(Q)$. The system $S_m = \{f_0, \cdots, f_m\}$ with $f_i \in C(Q)$ and $f_0(x) = 1$, $x \in Q$, is called a K-system of order m if, as $n \to \infty$, $L_n f_i \to f_i$ ($i = 0, \cdots, m$) implies $L_n f \to f$ for all $f \in C(Q)$. Among the results announced are the following. In order that there exist a K-system of finite order, it is necessary and sufficient that Q have finite dimension. A necessary and sufficient condition that S_m be a K-system is that for each $x \in Q$ there exist a unique positive linear functional, F, such that $F(f_i) = f_i(x)$ ($i = 0, \cdots, m$). An alternative condition is that the mapping T of Q onto a subset M of Euclidean m-space defined by $T(x) = (f_1(x), \cdots, f_m(x))$ be a homeomorphism and that each point of M be an extreme point of its convex hull. More detailed information is given concerning particular sets Q. *P. Civin* (Gainesville, Fla.)

8352:

Schue, John R. Hilbert space methods in the theory of Lie algebras. Trans. Amer. Math. Soc. **95** (1960), 69–80.

This paper considers infinite-dimensional Lie algebras L over the complex numbers for which the underlying linear space is a Hilbert space. It is assumed that for each $x \in L$ there is an $x^* \in L$ such that D_x is the adjoint operator of D_{x^*}, where $D_x y = [x, y]$. Such L are called L^*-algebras. An example is obtained by taking an (associative) H^*-algebra and defining $[x, y] = xy - yx$. The aim of this paper is to extend the usual theory of semi-simple Lie algebras to L^*-algebras.

L is defined to be semi-simple if $L = [L, L]$, and simple if there are no proper closed ideals. It is first shown that a

semi-simple L is uniquely a direct sum of simple closed ideals.

Now consider semi-simple L; a Cartan subalgebra is defined as a maximal self-adjoint abelian subalgebra. Denote it by H. Roots and root spaces are then defined as usual. L is said to have a Cartan decomposition (relative to H) if the sum of the root spaces is L. It is not settled whether every such L has a Cartan decomposition, except if L is an L^*-subalgebra of an H^*-algebra.

Now consider simple L. By looking at appropriate increasing sequences of simple finite-dimensional Lie algebras in L, and using the classification of finite-dimensional simple algebras, it is proved that L must be one of the following three non-isomorphic types: (A) all Hilbert-Schmidt operators; (B) all Hilbert-Schmidt operators T such that $T^*J=-JT$, for some conjugation J; (C) all Hilbert-Schmidt operators T such that $T^*J=-JT$, for some anti-conjugation J. It is shown that in each of these cases, derivations need not be inner.

W. Ambrose (Cambridge, Mass.)

8353:

Naimark, M. A. On the decomposition into irreducible representations of the tensor product of two representations of the supplementary series of the proper Lorentz group. Dokl. Akad. Nauk SSSR **130** (1960), 261–264 (Russian); translated as Soviet Math. Dok. **1**, 44–47.

This note announces the completion of results obtained earlier on the reduction into irreducible components of the tensor product of two irreducible unitary representations of the Lorentz group [same Dokl. **119** (1958), 872–875; **125** (1959), 1196–1199; Trudy Moskov Mat. Obšč. **8** (1959), 121–153; MR **20** #7228; **21** #5905; **22**#4966]. The case considered here is that in which both of the factors of the tensor product belong to the supplementary series. In this case the product is the continuous direct sum of representations of the principal series, and in some cases a factor from the supplementary series. The proof is sketched.

E. Hewitt (Seattle, Wash.)

8354:

Bade, W. G.; Curtis, P. C., Jr. Homomorphisms of commutative Banach algebras. Amer. J. Math. **82** (1960), 589–608.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative Banach algebras, and h be an arbitrary (not necessarily continuous) homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. This paper studies continuity properties of h which arise from the algebraic structure of $\mathfrak{A}$. The main results are grouped around the following topics. (1) The degree of discontinuity which h may have on the set of idempotents of $\mathfrak{A}$. (2) The localization of the discontinuity of h to a finite set of points of the structure-space of $\mathfrak{A}$ when $\mathfrak{A}$ is a regular algebra in the sense of Šilov. (3) The question of the existence of discontinuous homomorphisms on the algebra $C(\Omega)$ of all continuous functions on a compact Hausdorff space. (4) The construction of algebras which are not normed algebras under any norm. If h is a homomorphism of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$, then the function $x \to \|h(x)\|$ (where $x \in \mathfrak{A}$), is a multiplicative semi-norm on $\mathfrak{A}$. Conversely, every multiplicative semi-norm is the norm of a homomorphism; consequently, the results in the present paper could be stated equivalently in terms of continuity properties of multiplicative semi-norms on $\mathfrak{A}$.

Let h be a homomorphism defined on the algebra $\mathfrak{A}$. Now follows the key result of the paper. Let $(p_n)_n$ be a sequence of mutually orthogonal elements of $\mathfrak{A}$; if $(g_n)_n$ is an arbitrary sequence in $\mathfrak{A}$ such that $g_n p_n = g_n$ (for all values of n), then there exists a finite number K such that $\|h(g_n)\| \leq K\|g_n\| \, \|p_n\|$ for all values of n.

Let $\mathfrak{P}$ be the set of idempotents of $\mathfrak{A}$. If h is a homomorphism defined on $\mathfrak{A}$, then there exists a constant M such that $\|h(p)\| \leq M\|p\|^2$ for all p in $\mathfrak{P}$. This shows that, if $\mathfrak{P}$ is a bounded set in $\mathfrak{A}$, then $\mathfrak{P}$ remains bounded under any homomorphism. In section 3 the algebra $\mathfrak{A}$ is supposed to be commutative, semi-simple, with unit, and regular in the sense of Šilov. Again let h be a homomorphism defined on $\mathfrak{A}$; regarding $\mathfrak{A}$ as an algebra of continuous functions on its structure-space $\Phi_{\mathfrak{A}}$, it is shown that there is a finite set F of points of $\Phi_{\mathfrak{A}}$ such that h is continuous on the ideal of functions in $\mathfrak{A}$ which vanish identically in any neighborhood of F. Examples are given where F is not empty. Section 3 deals with the special case $A=C(\Omega)$. It is shown that any homomorphism h defined on $C(\Omega)$ has a decomposition $h=\mu+\lambda$, where μ is a continuous homomorphism of $C(\Omega)$, and where λ maps into the radical of the range of h; moreover, μ coincides with h on a dense sub-algebra, and

$$\overline{h(C(\Omega))} = \mu(C(\Omega)) \oplus \overline{\lambda(C(\Omega))},$$

the direct sum being topological. The existence of a discontinuous homomorphism on $C(\Omega)$ is shown to be equivalent to the existence of a non-trivial homomorphism of some maximal ideal of $C(\Omega)$ into a radical Banach algebra.

Section 5 deals with the non-normability of certain quotient algebras. Let $\mathfrak{A}$ be a semi-simple regular algebra with unit, $\varphi_0 \in \Phi_{\mathfrak{A}}$, and let $\mathfrak{I}(\varphi_0)$ be the ideal of all functions in $\mathfrak{A}$ which vanish in a neighborhood of φ_0; it is shown that the algebra $\mathfrak{A}/\mathfrak{I}(\varphi_0)$ is not normable in the case where φ_0 is the limit of a sequence of distinct points in $\Phi_{\mathfrak{A}}$. Section 6 contains a discussion of a Banach algebra due to C. Feldman [Proc. Amer. Math. Soc. **2** (1951), 771–777; MR **13**, 361] which shows that the main theorem of section 2 cannot be improved; it also provides an example of an algebra with one-dimensional radical which admits two inequivalent complete multiplicative norms. This shows that the theorem of Gelfand to the effect that semi-simple commutative Banach algebras have unique Banach algebra topologies cannot be generalized even to algebras with finite-dimensional radical. Let $\mathfrak{F}$ be the algebra of all absolutely convergent Fourier series; does $\mathfrak{F}$ have discontinuous homomorphisms? On the strength of some remarks added in proof, the authors reduce the preceding question to the (open) question of the existence of homomorphisms of $\mathfrak{F}$ into a radical Banach algebra.

G. L. Krabbe (New Haven, Conn.)

8355:

Waelbroeck, L. Étude spectrale des algèbres complètes. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) **31** (1960), no. 7, 142 pp.

In n-space C^n let $\delta_0(s)=(1+|s|^2)^{-1/2}$, where $s=(s_1, s_2, \cdots, s_n)$ and $|s|^2=\sum_{i=1}^n |s_i|^2$. For an algebra A (with identity) in which there is defined a notion of boundedness, let $\Theta(s;\delta;A)$ be the family of locally bounded A-valued functions u on C^n for which $\delta^N u$ is bounded for large enough N. For $a=(a_1, a_2, \cdots, a_n)$ an n-tuple of elements

in A, $\Delta(a; A)$ (spectrum of a) is defined as the set of bounded non-negative functions δ for which there exist functions $u_1, u_2, \cdots, u_n, y_0$ in $\Theta(s; \delta_0; A)$ satisfying (*) $\sum_{i=1}^n (a_i - s_i)u_i(s) + \delta(s)y_0(s) \equiv 1$. For $\delta \in \Delta(a; A)$ let $\mathfrak{O}(s; \delta; A)$ be the family of A-valued functions f holomorphic in $\{s \mid \delta(s) > 0\}$ and such that $\delta^N f$ is bounded for large N. The exterior form $\omega = [(n+k)!/k!]y^k du_1 \wedge \cdots \wedge du_n \wedge ds_1 \wedge \cdots \wedge ds_n$, where $y = \delta y_0$ (see (*)) is in a cohomology class (in the space of forms with coefficients divisible by δ^r) that is independent of $(u_1, u_2, \cdots, u_n, y_0)$ when k is large, and the same is true of $f\omega$ for any f in $\mathfrak{O}(s; \delta; A)$. Hence $\mathrm{S}(f; a; A) \equiv (2\pi i)^{-n} \int f(s)\omega$ is independent of $(u_1, u_2, \cdots, u_n, y_0)$ for large k. The prime consequence of this is the theorem: The mapping $\theta: f \to \mathrm{S}(f; a; A)$ is a homomorphism of $\mathfrak{O}(s; \delta; A)$ for which $\theta(s_i) = a_i$.

Clearly the above constitutes a vast generalization of known results in Banach algebras [I. Gel'fand, Mat. Sb. (N.S.) **9** (**51**) (1941), 51–66; MR **3**, 51; R. Arens and A. Calderón, Ann. of Math. (2) **62** (1955), 204–216; MR **17**, 177]. The author devotes 137 pages of text to the task of establishing (1) a relativized version of the theorem quoted and (2) "applications", i.e., discussions of special cases. An idea of the range of material developed may be gained from a consideration of the chapter headings: I. Les espaces à bornés complets; II. Fonctions tempérées; III. Formes extérieures et cohomologie; IV. Le spectre; V. Une classe de cohomologie; VI. Le calcul symbolique; VII. Applications.

An open problem is offered to indicate one possible line of further research. Let $\mathrm{S}(f; a; A)$ be denoted by $f[a]$. Then, for f, g in $\mathfrak{O}(s; \delta; A)$, is $f[g(a)] = f(g)[a]$ a valid equation? *B. R. Gelbaum* (Princeton, N.J.)

8356:

Foguel, S. R. Normal operators of finite multiplicity. Comm. Pure Appl. Math. **11** (1958), 297–313.

The structure of the algebra $\mathfrak{A}$ of bounded operators on Hilbert space H which commute with a given normal operator S of finite multiplicity n is studied, in relation to the spectral operators of Dunford [Pacific J. Math. **4** (1954), 321–354; MR **16**, 142], as follows. By assumption, S may be regarded as the multiplication $S\{f_1(\lambda), \cdots, f_n(\lambda)\} = \{\lambda f_1(\lambda), \cdots, \lambda f_n(\lambda)\}$ where $f_i \in L_2(\mu_i)$, μ_i being the restriction to a subset e_i of a Borel measure supported on the spectrum of S [Dunford and Schwarz, #8302], and it follows that every $A \in \mathfrak{A}$ has a unique matricial representation $A = (a_{ij}(\lambda))$, with a_{ij} supported on $e_i \cap e_j$, $a_{ij} \in L_\infty(\mu)$. By essentially analytical study of the a_{ij}, strong convergence in $\mathfrak{A}$ is shown equivalent to convergence in measure of the a_{ij}, which implies the corollary that adjunction is strongly continuous in $\mathfrak{A}$; and a Jordan canonical form $A(\lambda) = \sum z_i(\lambda)\varepsilon_i(\lambda) + N(\lambda)$ is derived, where $A(\lambda) = (a_{ij}(\lambda))$, the matrices $\varepsilon_i(\lambda)$ are idempotents adding to I, $N(\lambda)$ is a nilpotent matrix, and the z_i are essentially bounded functions. It follows that every A in $\mathfrak{A}$ is the strong limit of spectral operators and that the generalized nilpotents of $\mathfrak{A}$ are already nilpotent. Characterizations of spectral and compact elements of $\mathfrak{A}$ are given in terms of the Jordan form, and in particular it is shown that a compact $A \in \mathfrak{A}$ has finite rank and is spectral if S has finitely many eigenvalues and is 0 if S has no eigenvalues.

M. Schreiber (Ithaca, N.Y.)

8357:

Minakshisundaram, S. Hilbert algebras. Proc. Internat. Congress Math. 1958, pp. 407–411. Cambridge Univ. Press, New York, 1960.

Expository description, without proofs, of Hilbert algebras. Subjects covered: axioms, abelian algebras, simple algebras, and decomposition of non-abelian and non-simple algebras with respect to the character spaces of the centers. *M. Nakamura* (Asiya)

8358:

Tomita, Minoru. Spectral theory of operator algebras. I. Math. J. Okayama Univ. **9** (1959/60), 63–98.

The author treats states of C^*-algebras and the von Neumann reduction theory of operator rings along lines similar to those previously employed by the reviewer (as regards states) and by Godement (as regards fields), but with some improvements and extensions. The major novelty is the introduction and study of the Hilbert space of square-integrable functionals with respect to a state of a C^*-algebra. This works out neatly, and provides a basis for a more refined treatment of reduction theory. He treats also quotient algebras of C^*-algebras, achieving an extension of Kadison's theorem that for a pure state the representation pre-Hilbert space associated is automatically complete. His extension is less simply stated, involving the notion of a regular projection, which he introduces and treats. In this connection he improves the von Neumann-Kaplansky density theorem and makes a constructive correction of an assertion of Kaplansky on the regularity of certain projections. The separability of the C^*-algebra remains necessary for the achievement of elementarity almost everywhere of the infinitesimal constituents in reduction theory. *I. E. Segal* (Cambridge, Mass.)

8359:

Kovács, I. Un complément à la théorie de l' "intégration non commutative". Acta Sci. Math. Szeged **21** (1960), 7–11.

Soit M un anneau d'opérateurs dans un espace hilbertien complexe; on suppose que M est de genre dénombrable (ceci veut dire que toute famille de projecteurs de M non nuls et deux à deux orthogonaux est dénombrable). Soit M^+ l'ensemble des éléments hermitiens ≥ 0 de M. Une forme linéaire ρ sur M est dite positive si l'on a $\rho(T) \geq 0$ pour tout $T \in M^+$. Soit G l'ensemble des formes linéaires positives ρ telles que, pour tout ensemble filtrant croissant $F \subset M^+$ de borne supérieure $T \in M^+$, la borne supérieure de $\rho(F)$ est égale à $\rho(T)$.

On suppose que M ne contient que des projecteurs finis. On peut donc construire une théorie de l'intégration pour les opérateurs de M [cf. J. Dixmier, Bull. Soc. Math. France **81** (1953), 9–39; MR **15**, 539; I. E. Segal, Ann. of Math. (2) **57** (1953), 401–457; MR **14**, 991]. L'auteur démontre le théorème suivant. Soit $(\rho_n)_{n \geq 1}$ une suite de formes linéaires qui appartiennent à G; si la limite $\rho(T) = \lim_{n\to\infty} \rho_n(T)$ existe pour tout $T \in M$, alors la forme linéaire $T \to \rho(T)$ appartient à G. Ce théorème est un pendant "non commutatif" du théorème de Vitali-Hahn-Saks. Corollaire: L'espace $(L^1)^+$ est faiblement complet.

G. L. Krabbe (New Haven, Conn.)

8360:

Rolewicz, S. Remarks on functions with derivative zero. Wiadom. Mat. (2) **3**, 127–128 (1959). (Polish)

Let X be a linear metric space with metric ρ. A mapping $x=x(t)$ of $0<t<1$ into X is said to be differentiable if for all $0<t<1$, $\lim_{h\to 0} h^{-1}(x(t+h)-x(t))=x'(t)$ exists in X. If x is differentiable and for all $0<t<1$, $x'(t)=0$, then $x(t)$ is not necessarily equal to a constant function. However, if X admits a total family f_α of continuous linear functionals, then $x'(t)=0$ for all $0<t<1$ implies that $x(t)$ is equal to a constant, since in that case all functions $f_\alpha(x(t))$ are differentiable and have their derivatives equal to zero. S. Mazur has proposed the following question: Does the space X have the same property if X has a total family of continuous, but not necessarily linear, real-valued mappings? In this note the author shows that the answer to this question is in general no. To this end the author takes for X the metric space of all measurable functions $x=x(\tau)$ $0<\tau<1$. For any $x=x(\tau)$, let $x(t)=x(\tau)$ if $\tau\leqq t$ and let $x(t)=0$ if $t<\tau$. Then

$$\rho\left(\frac{x(t+h)-x(t)}{h}, 0\right) \leqq |h|,$$

where

$$\rho(x, y) = \int_0^1 \frac{|x(\tau)-y(\tau)|}{1+|x(\tau)-y(\tau)|}\,d\tau.$$

Hence $x'(t)=0$ for all $0<t<1$, but x is not a constant. However, the family consisting of the mapping $\rho(x, 0)$ is a total family on X. Note in passing that this linear metric space X does not have a total family of continuous linear functionals. *W. A. J. Luxemburg* (Pasadena, Calif.)

8361:

Yamamuro, Sadayuki. On the theory of non-linear operators. Proc. Japan Acad. **36** (1960), 305–309.

Let R denote a reflexive Banach space, $\bar{R}$ its dual, K a continuous linear map on R into $\bar{R}$ with $\langle x, Kx\rangle\geqq 0$ ($x\in R$), F a convex, differentiable real function on R with $F(0)=0$, $F(x)\to+\infty$ for $\|x\|\to+\infty$. It is shown by a variational method that for each $r>0$, there exists x_r such that $Kx_r=\lambda_r F'(x_r)$ for suitable $\lambda_r>0$, with $F(x_r)=r$. [Cf. the reviewer's work in Pacific J. Math. **9** (1959), 847–860; **10** (1960), 1479; MR **22**#1827.] This result is applied to prove the existence of eigenvalues for $\lambda\varphi=H\varphi$, where H is an integral operator of Hammerstein type subject to suitable additional conditions. {Reviewer's remark: Assumption R1, p. 306, is too strong and excludes the examples mentioned by the author.}

H. H. Schaefer (Ann Arbor, Mich.)

8362:

Trenogin, V. A. Branching of solutions of non-linear equations in Banach space. Uspehi Mat. Nauk **13** (1958), no. 4 (82), 197–203. (Russian)

The author considers the solutions of an equation (*) $F(x, y)=0$, where the arguments and values lie in real Banach spaces. It is supposed that (x_0, y_0) is a solution of (*), that F has continuous second Fréchet derivatives near (x_0, y_0) and that $(\partial F/\partial y)(x_0, y_0)$ is a non-invertible linear operator for which the Fredholm theory holds. Under stated hypotheses it is seen that there is a star-shaped neighborhood of x_0 such that in this neighborhood certain of the solutions of (*) are given by n-quadratic equations in n unknowns. Expansions in fractional powers are briefly considered. The main ideas go back to E. Schmidt, A. Lyapunov and L. Lichtenstein.

R. G. Bartle (Urbana, Ill.)

8363:

Trenogin, V. A. Branching equation and Newton diagram. Dokl. Akad. Nauk SSSR **131** (1960), 1032–1035 (Russian); translated as Soviet Math. Dokl. **1**, 388–391.

Let E, E_1, and E_2 be real or complex Banach spaces and let F map $E\times E_1$ into E_2. The author is concerned with the equation $F(x, y)=0$ in the neighborhood of a solution (x_0, y_0). It is assumed that the Fréchet derivative $(\partial F/\partial y)(x_0, y_0)$ is a Fredholm operator in the sense that its range is closed and it has n-dimensional null space and m-dimensional deficiency (but that possibly $n\neq m$). The branching equation ("Verzweigungsgleichung") of A. Lyapunov and E. Schmidt is derived for this case in the standard way. It is suggested that the Newton diagram can be used to obtain at least the principal term of the local solutions without constructing the branching equation. Under appropriate hypotheses one obtains the solutions as fractional power series. {In the Definition on p. 388 of the translation "domain of definition" should read "range".} *R. G. Bartle* (Urbana, Ill.)

8364:

Kivistik, L. A. A modification of the iterative method with minimal residuals for the solution of non-linear operator equations. Dokl. Akad. Nauk SSSR **136** (1961), 22–25 (Russian); translated as Soviet Math. Dokl. **2**, 13–16.

Iterative methods of solution of $P(x)=0$ are considered, where x is an element of a Hilbert space and P an operator from this space into the same space. The existence of a solution x^* and the convergence of the sequence x_n to this solution are considered, where x_0 is an initial approximate solution, and $x_{n+1}=x_n+\varepsilon_n P(x_n)$,

$$\varepsilon_n = -(P(x_n), P'(x_n)y_n)/\|P'(x_n)y_n\|^2 \quad (n = 0, 1, \cdots).$$

F. Goodspeed (Quebec)

8365:

Granas, A. Theorem on antipodes and theorems on fixed points for a certain class of multi-valued mappings in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 271–275. (Russian summary, unbound insert)

Let Φ be an upper semi-continuous point-to-convex-set map on a subset P to the Banach space E, where it is understood that $\Phi(P)$ is compact. Write $\varphi x=x-\Phi(x)$. The Leray-Schauder type of index exists and, with P taken as the unit ball or the unit sphere or as a convex compact set, many of the familiar fixed point theorems, valid for Φ a point-point map, are valid.

D. G. Bourgin (Urbana, Ill.)

8366:

Gel'fand, I. M. On some problems of functional analysis. Amer. Math. Soc. Transl. (2) **16** (1960), 315–324.

Translation of Uspehi Mat. Nauk **11** (1956), no. 6 (72), 3–12 [MR **19**, 293].

8367:
Berezin, F. A.; Gel'fand, I. M.; Graev, M. I.; Naimark, M. A. Group representations. Amer. Math. Soc. Transl. (2) **16** (1960), 325–353.
Translation of Uspehi Mat. Nauk **11** (1956), no. 6 (72), 13–40 [MR **19**, 662].

8368:
Berezanskiĭ, Yu. M.; Kreĭn, S. G. Hypercomplex systems with continuous basis. Amer. Math. Soc. Transl. (2) **16** (1960), 358–364.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 147–152 [MR **19**, 154].

8369:
Borisovič, Yu. G. On the critical values of some functionals in Banach spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 369–373.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 157–160 [MR **19**, 755].

8370:
Vaĭnberg, M. M. Some problems of functional analysis and the variational methods of studying nonlinear equations. Amer. Math. Soc. Transl. (2) **16** (1960), 375–378.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 162–165 [MR **19**, 48].

8371:
Vertgeĭm, B. A. On some methods of the approximate solution of nonlinear functional equations in Banach spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 378–382.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 166–169 [MR **19**, 267].

8372:
Vulih, B. Z. Application of the theory of partially ordered spaces to the study of self-adjoint operators in Hilbert space. Amer. Math. Soc. Transl. (2) **16** (1960), 382–385.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 169–172 [MR **19**, 47].

8373:
Gavurin, M. K. Approximate determination of proper values and perturbation theory. Amer. Math. Soc. Transl. (2) **16** (1960), 385–388.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 173–175 [MR **19**, 756].

8374:
Gohberg, I. C. On the index, zero-elements and kernel elements of unbounded operators. Amer. Math. Soc. Transl. (2) **16** (1960), 391–392.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 177–179 [MR **19**, 45].

8375:
Graev, M. I. Unitary representations of real simple Lie groups. Amer. Math. Soc. Transl. (2) **16** (1960), 393–396.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 179–182 [MR **19**, 431].

8376:
Daleckiĭ, Yu. L. Integration and differentiation of functions of hermitian operators depending on a parameter. Amer. Math. Soc. Transl. (2) **16** (1960), 396–400.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 182–186 [MR **19**, 155].

8377:
Zuhovickiĭ, S. I.; Stečkin, S. B. On the approximation of abstract functions. Amer. Math. Soc. Transl. (2) **16** (1960), 400–406.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 187–191 [MR **19**, 267].

8378:
Kaazik, Yu. Ya. On the approximate solution of non-linear operator equations by iterative methods. Amer. Math. Soc. Transl. (2) **16** (1960), 410–413.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 195–199 [MR **19**, 687].

8379:
Krasnosel'skiĭ, M. A. Investigation of the spectrum of a non-linear operator in a neighborhood of a branch point, and application to the problem of longitudinal bending of a compressed rod. Amer. Math. Soc. Transl. (2) **16** (1960), 418–423.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 203–208 [MR **19**, 45].

8380:
Ladyženskiĭ, L. A. On non-linear equations with positive non-linearity. Amer. Math. Soc. Transl. (2) **16** (1960), 426–427.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 211–212 [MR **19**, 45].

8381:
Mel'nik, S. I. The principle of St. Venant and oscillating functions. Amer. Math. Soc. Transl. (2) **16** (1960), 434–437.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 218–222 [MR **18**, 938].

8382:
Mil'man, D. P. Some theorems of non-linear functional analysis and their application in the theory of local groups. Amer. Math. Soc. Transl. (2) **16** (1960), 437–442.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 222–226 [MR **20** #2636].

8383:
Pinsker, A. G. Structural characterization of functional spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 442–445.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 226–229 [MR **19**, 7].

8384:
Rutickiĭ, Ya. B. On a class of Banach spaces. Amer. Math. Soc. Transl. (2) **16** (1960), 445–450.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 230–234 [MR **19**, 45].

8385:
Rutman, M. A. Operator equations in semi-ordered spaces and some qualitative theorems for linear partial differential equations. Amer. Math. Soc. Transl. (2) **16** (1960), 450–454.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 234–238 [MR **21** #5137].

8386:
Samokiš, B. A. The investigation of the rapidity of convergence of the method of steepest descent. Amer. Math. Soc. Transl. (2) **16** (1960), 454–456.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 238–240 [MR **19**, 322].

8387:
Štraus, A. V. Generalized resolvents of symmetric operators and eigenfunction expansions for a certain class of boundary-value problems. Amer. Math. Soc. Transl. (2) **16** (1960), 462–464.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 251–253 [MR **19**, 47].

8388:
Šilov, G. E. On certain problems of the general theory of commutative normed rings. Amer. Math. Soc. Transl. (2) **16** (1960), 471–475.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 246–249 [MR **18**, 912].

8389:
Gel'fand, I. M. On the subrings of a ring of continuous functions. Amer. Math. Soc. Transl. (2) **16** (1960), 477–479.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 249–251 [MR **18**, 913].

CALCULUS OF VARIATIONS

See also 8169.

8390:
Èl'sgol'c, L. È. Variational problems with a delayed argument. Amer. Math. Soc. Transl. (2) **16** (1960), 468–469.
Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 257–258; a brief discussion of variational problems of the form

$$\int_{t_0}^{t_1} F(t, x(t), x(t-\tau), \dot{x}(t), \dot{x}(t-\tau))\, dt, \quad \tau > 0.$$

8391:
Reifenberg, E. R. Solution of the Plateau problem for m-dimensional surfaces of varying topological type. Bull. Amer. Math. Soc. **66** (1960), 312–313.
This is an announcement of results of the author's paper in Acta Math. **104** (1960), 1–92 [MR **22** #4972].
W. H. Flemming (Providence, R.I.)

GEOMETRY

See also 8005, 8058, 8066, 8438, 8462, B9314.

8392:
Birkhoff, George David; Beatley, Ralph. ★Basic geometry. 3rd ed. Chelsea Publishing Company, New York, 1959. 294 pp.
The necessity to rewrite Euclidean geometry for high schools in the light of 20th century mathematical knowledge and technique led the two authors to publish in 1933 an experimental edition of this book [*Geometry*, Spaulding-Mass, Boston, 1933], and test its pedagogical value for several years at the Newton, Massachusetts, High School, before publishing it as *Basic geometry* [Scott, Foresman, Chicago, Ill., 1940; 1941; 1959]. It attempts to deduce plane Euclidean geometry on the basis of only five principles and adding seven theorems (one of them the Pythagorean theorem); some of them, though provable, are introduced as obvious and proven later, after skill and insight into the exact procedure of demonstration has been acquired. The five basic geometrical principles are quite powerful, because they introduce from the beginning Descartes' arithmetisation (to each segment is attached a real number); the fact that their full utilization silently accepts a series of axioms on numbers is mentioned and they are enumerated at the end of the book to the teacher's or novice's satisfaction. The following (p. 199) is an example of the pedagogical procedure: Area assumption 1: Every polygon has a number, called its area; equal polygons have equal areas; the area of a polygon is equal to the sum of its constituent polygons. Area assumption 2: The area of a rectangle is the product of its length times its width. Later (p. 222), the exact procedure, avoiding assumptions 1 and 2, is sketched in a one-page account, citing J. Hadamard [*Leçons de géométrie élémentaire Vol. I*, Librairie Armand Colin, Paris, 1911]. The concept of limit is introduced, when computing $p_4, p_8, p_{16}, \cdots$, the perimeters of inscribed regular polygons, arriving thus at an evaluation of π. Throughout the book, too elaborate proofs or subtle considerations are omitted. Exercises, in consistent succession, contain essential theorems, and thus shorten the text. To begin with and interspersed in the text are references to real life, illustrating the application of the abstract concepts introduced, and (most commendable) leading to the realization that geometry is an example, par excellence, of a faultless logical system, used only by approximation in different important aspects of scientific and daily life.
S. R. Struik (Cambridge, Mass.)

8393:

★**Grundzüge der Mathematik. Bd. II: Geometrie.** Herausgegeben von H. Behnke, F. Bachmann, K. Fladt, W. Süss. Vandenhoeck & Ruprecht, Göttingen, 1960. xvi+646 pp. DM 58.00.

This is the second of a projected four-volume series [for Vol. I, on foundations, arithmetic and algebra, see MR **20** #4472] of expository articles on selected topics, addressed primarily to the gymnasium student and designed to acquaint him with modern viewpoints and developments. The articles are well illustrated and supplied with references to the literature (mainly German), both current and "classical". This volume is dedicated posthumously to the last-named editor, Wilhelm Süss. The articles are as follows (in most cases they are confined to dimension $\leqq 3$). 1. H. Freudenthal, A. Baur: Phenomenological geometry [i.e., intuitive background and heuristics]. 2. W. Klingenberg, A. Baur: Axiomatic foundations of euclidean and non-euclidean geometry [Hilbert axioms]. 3. F. Bachmann, H. Wolff, A. Baur: Reflections [as a basis for axioms]. 4. R. Lingenberg, A. Baur: The synthetic and analytic viewpoint [affine and projective geometry; coordinatization]. 5. W. Breidenbach, W. Süss: Geometric constructions [by various means: straight-edge alone, higher order constructions, etc.]. 6. J. Gerretsen, P. Vredenduin: Polygons and polyhedra. 7. H. Gericke, F. Raith: Vectors and trigonometry. 8. G. Pickert, R. Stender, M. Hellwich: Projective, affine and metric geometry [Mappings, forms of degree 2]. 9. W. Burau, A. Baur: Algebraic geometry. 10. W. Süss, K. Fladt: Klein's Erlanger Programm [including the geometries of Möbius, Lie, Laguerre, etc.]. 10a. H. Freudenthal, H.-G. Steiner: Group theory and geometry [abstract transformation groups, Raumproblem]. 11. F. Hohenberg, J. Tschupik: Descriptive geometry. 12a. W. Süss, H. Gericke, K. H. Berger: Curves and surfaces—general theory [metric differential geometry, including tensor analysis and Riemannian geometry]. 12b. W. Süss, U. Viet, K. H. Berger: Curves and surfaces—convex figures [in the plane]. 13. K. H. Weise, H. Noack: Selected topics in topology [2-dimensional surfaces and elementary homology theory; curves and point-set theory; general spaces].

8394:

Jaeger, Arno. ★**Introduction to analytic geometry and linear algebra.** Holt, Rinehart and Winston, Inc., New York, 1960. xii+305 pp. $6.00.

The book contains a remarkable and, in the opinion of the reviewer, successful attempt at revising the teaching of analytic geometry and linear algebra for the undergraduate in a manner which anticipates the later needs of the student of mathematics. On the one hand, the language is abstract and precise; on the other hand, the book offers substantial knowledge and not merely a vocabulary in every chapter. The word "geometry" deserves its place in the title; geometrical theorems like those due to Ceva or to Desargues appear early (pp. 83, 84) in the text. The book begins with an explanation of the difficulties of mathematical language. The author writes out formulas in detail first before using the more economical concise notations. Examples and exercises on various levels of difficulty abound. At the end, there is a "Guide to further reading", an index of examples, of symbols and of terms. Special mention should be given to the chapter on linear programming.

Table of contents: Part I (Foundations). 1: Introduction. 2: Elementary set theory. 3: Translations. 4: Composition laws and groups. 5: Vector spaces. 6: The idea of analytic geometry. 7: Lines and planes. Part II (Linear geometry and algebra). 8: Linear systems. 9: Dimensions and bases of a vector space. 10: Positive solutions of systems of linear equations. 11: Linear programming. 12: The calculus of matrices. 13: Special matrices. 14: Linear mappings. Part III (Multilinear geometry and algebra). 15: Length and angle. 16: Euclidean and unitary vector spaces. 17: A recursive definition of the determinant. 18: Basic properties of the determinant. 19: Orientation, area, volume, vector product. Part IV (Quadratic geometry and algebra). 20: Circles and spheres. 21: The classical conic sections. 22: Reduction of quadratic polynomials. 23: Classification of conics and quadrics.

W. Magnus (New York)

8395:

Morin, U. Geometria elementare e teoria dei gruppi. Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 101–113. Edizioni Cremonese, Rome, 1960.

(1) Exposition of a few notions of plane reflexion geometry. (2) Geometry of the straight line with congruence and order axioms. The author formulates a problem which interpreted in group theory would mean the problem of the existence of dense ordered non-commutative groups. Such an example, however, can be found in Hilbert's *Grundlagen der Geometrie* [8te Aufl., Teubner, Stuttgart, 1956; MR **18**, 227].

H. Freudenthal (Utrecht)

8396:

Hofmann, Joseph Ehrenfried. Über die Quadrisectio trianguli. Math. Z. **74** (1960), 105–118.

The history of the problem to divide a triangle in four equal parts by two orthogonal lines. Huygens learnt the problems from M. de Maubuisson. After Huygens, Jakob Bernoulli and Euler tackled it. The author adds a remark of his own on the construction of the solutions and a remark of Speiser's on the number of the solutions.

H. Freudenthal (Utrecht)

8397:

Busulini, Franca. Sui punti notevoli del triangolo. Period. Mat. (4) **37** (1959), 301–307.

The existence of (1) the circumcenter, (2) the orthocenter, and (3) the barycenter of a triangle, is proved to be independent of Euclid's parallel axiom: in case (1), if the greatest angle is smaller than the sum of the two others; in case (2), if the triangle has acute angles or its greatest angle is not greater than 4/3 of a right angle; and always in case (3). Case (3) is demonstrated in a purely geometric way.

S. R. Struik (Cambridge, Mass.)

8398:

Kooistra, R. Einige Ungleichungen. Elem. Math. **15** (1960), 79–80.

Let F be the area of a triangle ABC and F_p the area of the triangle whose vertices are the feet of the perpendiculars from P to the sides of ABC. The author proves

that if P is interior to the circumcircle C of the triangle ABC, then $F_P \leq F/4$, the equality holding if P is the centre of C. By specializing the point P, other curious inequalities are obtained. *L. A. Santaló* (Buenos Aires)

8399:

Thébault, Victor. Sur la surface de Sartiaux. Mathesis **69** (1960), 126–131.

Author's summary: "Cette note réunit des propriétés d'une surface S dont certaines ont trait aux associés harmoniques du point dont les distances aux faces d'un tétraèdre sont proportionnelles aux aires de ces faces (premier point de Lemoine)."

8400:

Schubart, H. Geometrische Konfigurationen und komplexe Zahlen. Math. Naturwiss. Unterricht **13** (1960/61), 345–348.

Let z_1, z_2, z_3 be three points (complex numbers) in the Gaussian plane, and let $\eta = \exp \pi i/3$. A rotation of the plane counterclockwise through the angle $\pi/3$ about the point O takes z_1, z_2, z_3 into $z_1\eta$, $z_2\eta$, $z_3\eta$. Let I, II, III, I′, II′, III′ be the mid-points of the segments $(z_1\eta, z_2)$, $(z_2\eta, z_3)$, $(z_3\eta, z_1)$, $(z_1, z_2\eta)$, $(z_3, z_1\eta)$, $(z_2, z_3\eta)$, respectively. Using elementary methods, the author proves a number of theorems regarding configurations involving these points, e.g.: the triangles I, II, III and I′, II′, III′ are equilateral; I I′ = II II′ = III III′ $= \frac{1}{2}(z_1 - z_2)$, etc.; if z_1 is the mid-point of (z_2, z_3), then I, I′, II, II′, III, III′ are the vertices of a regular hexagon.

W. Moser (Winnipeg, Man.)

8401:

ten Doesschate, G. Perception of parallelism. Euclides (Groningen) **35** (1959/60), 127–134. (Dutch)

Historical exposition of this aspect of geometric optics.

8402a:

Ganassini, M. L. ★Applicazioni di geometria descrittiva: Prospettiva. Università degli Studi di Roma, Facoltà di Architettura, Rome, 1959. 97 pp. L. 1500.

8402b:

Ganassini, M. L. ★Applicazioni di geometria descrittiva: Teoria delle ombre. Università degli Studi di Roma, Facoltà di Architettura, Rome, 1960. 140 pp. L. 2800.

The two books belong into the field of architectural graphics or architectural drawing. For a descriptive-geometer they are interesting from the viewpoint of application of descriptive geometry in dealing with perspective and shadows. For an architect they are interesting since they give a more theoretical background of the procedures involved than is usually done. The usefulness of the books would be greatly enhanced by adding an index or at least a table of contents.

D. Mazkewitsch (Cincinnati, Ohio)

8403:

Kárteszi, Francesco. Due piccoli contributi alla geometria elementare. Boll. Un. Mat. Ital. (3) **15** (1960), 54–58.

The author considers the regular tessellations {4, 4}, {3, 6}, {6, 3} of the Euclidean plane [cf. Coxeter, *Regular polytopes*, Methuen, London, 1948; MR **10**, 261; p. 59]. He gives an elementary proof that {6, 3} has no four vertices forming a square, and that {4, 4} has no three vertices forming an equilateral triangle (i.e., that the vertices of an equilateral triangle cannot all have integral Cartesian coordinates). *H. S. M. Coxeter* (Toronto)

8404:

De Cicco, John. The geometry of the z-plane based on a quadratic extension Γ of a field K^*. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 91–119.

Let Γ be a quadratic ring over the field K [De Cicco, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 225–242; MR **21** #3444]; a binary element of Γ is an element $z = x + iy$ where x and y are in K, i is not in K, but $i^2 = a + ib$ where a and b are two fixed elements of K. A two-dimensional vector space over K in which every point is represented by a binary element of Γ is called the z-plane. The z-plane is called elliptic, parabolic or hyperbolic, depending on whether the quadratic equation $i^2 = a + ib$ has none, one, or two solutions in K. Various geometric concepts such as similitudes and inner products for the different classes of z-planes are discussed in detail.

E. H. Batho (Durham, N.H.)

8405:

Gallarati, Dionisio. Una proprietà caratteristica della varietà di C. Segre prodotto di due spazi lineari. Rend. Mat. e Appl. (5) **17** (1958), 439–452.

The Segre variety $S_p \times S_q$ is the only C^2-differentiable variety of dimension $p+q$ in S_{pq+p+q} with the property that its tangent spaces S_{p+q} all meet $p+2$ of the S_q, any $p+1$ of which are independent; the points of intersection with each S_q forming a variety of dimension q, while the tangent spaces S_{p+q} do not all meet a S_{pq+p-1} in S_p.

J. A. Todd (Cambridge, England)

8406:

Marchionna, E. Sur la postulation des variétés algébriques et questions connexes. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 43–64. Centre Belge Rech. Math., Louvain, 1960.

A lecture based on results, partly due to the author, already expounded in the treatise by F. Severi: *Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica*, Vol. 3, [Edizioni Cremonese, Rome, 1959; MR **21** #3409]. A complete account is in course of publication in the Ann. Mat. Pura Appl. *L. Roth* (London)

8407:

Turri, Tullio. Le trasformazioni birazionali del piano aventi curva di punti uniti. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 121–129.

The author considers automorphisms of the projective plane which have a curve of fixed points. His results should be compared with those of G. Pompilj [Rend. Sem. Mat. Roma (4) **2** (1938), 47–87] and L. Derwidué [Mém. Soc. Roy. Sci. Liége (4) **7** (1946), 197–366; and Bull. Soc. Roy. Sci. Liége **16** (1947), 31–37; MR **9**, 527, 528].

A. Gutwirth (Haifa)

8408:

Keller, Ott-Heinrich. Zur Theorie der ebenen birationalen Berührungstransformationen. III. Der Grad der Bildkurven. Math. Ann. **139**, 239–254 (1960).

This is continuation of two papers by the same author on the same subject [Math. Ann. **120** (1949), 650–675; **121** (1950), 467–495; MR **10**, 736; **12**, 199]. Given two projective planes e, E, let x be a point $\in e$, X a point $\in E$, $F(X, x)=0$ the "aequatio directrix" of a birational contact transformation, of degree M in X and m in x. The problem considered in this paper is to find the order N of the transform of an algebraic curve $f\in e$ of order n. The result is of the form $N=Mn(n+2m-3)-N_1$. The integer N_1 (whose exact expression is given at the beginning of page 253) depends on: the base-points of F in e, their multiplicities and the multiplicities of f at them; the multiple points of f which are not base-points and their satellites; the degrees of the correspondence between double points and cusps of the curves $F=0$ in E and e; the degrees of the fundamental curves of the first kind in E and e (as defined in the previous papers), and the orders of contact of f and $F=0$ with them; the degrees of the fundamental curves of the second kind and the behaviour of f with respect to them. It is further proved that the system of curves defined by the fundamental elements is irreducible. *E. Bompiani* (Rome)

8409:

Manara, Carlo Felice. Questions d'existence de variétés algébriques. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 95–106. Centre Belge Rech. Math., Louvain, 1960.

A summary of results, by twelve different authors, concerning plane curves with assigned numbers of nodes and cusps, and the so-called multiple planes and their extensions, with special reference to the existence questions. *B. Segre* (Rome)

8410:

Laurenti, F. Sopra una proprietà dell'ipocicloide tricuspidata. Period. Mat. (4) **38** (1960), 155–158.

Dall'introduzione dell'A.: "Mi propongo di mostrare che tale curva è l'inviluppo degli assi delle parabole che passano per un punto dato, vi hanno una stessa tangente ed uno stesso raggio di curvatura".

8411:

Dedò, M. Sopra una proprietà della epicicloide tricuspidata. Period. Mat. (4) **38** (1960), 316–319.

Un'altra dimostrazione del risultato descritto nella recensione precedente. Si tratta comunque di ipocicloidi, non epicicloidi.

8412:

Godeaux, Lucien. Une propriété des doubles sixains d'une surface cubique. Mathesis **69** (1960), 5–7.

"Dans notre *Introduction à la géométrie supérieure* [Masson & Cie, Paris, 1946; MR **14**, 401], nous avons montré qu'une surface cubique générale étant donnée, il existe une surface du quatrième ordre coupant cette surface suivant les douze droites d'un double sixain. Dans cette note, nous nous proposons de donner une construction d'une telle surface."

8413:

Chisini, Oscar. Dimostrazione della rappresentabilità di una falda di superficie mediante serie procedenti per le potenze fratte di due variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 439–441.

A simplified version of a note of 1920: Given an algebraic surface $F(x, y, z)=0$, the author shows that, in the neighbourhood of the origin, the function $z(x, y)$ which represents one sheet of the surface can be expressed as a double series in fractional powers of x and y. *L. Roth* (London)

8414:

Godeaux, Lucien. Quelques résultats sur les surfaces de genres zéro possédant des courbes bicanoniques irréductibles. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 147–159. Centre Belge Rech. Math., Louvain, 1960.

The author sums up the results he obtained on algebraic surfaces with $p_a=p_g=0$ and $P_2>0$, since the appearance of his monograph *Les surfaces algébriques non rationnelles de genres arithmétique et géométrique nuls* [Hermann, Paris, 1934]. The main tool used here is the study of reducible m-canonical curves, $m>2$. *A. Gutwirth* (Haifa)

8415:

Burniat, Pol. Surfaces algébriques régulières de genre géométrique $p_g=0, 1, 2, 3$ et de genre linéaire $p^{(1)}=3, 4, \cdots, 8p_g+7$. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 129–145. Centre Belge Rech. Math., Louvain, 1960.

Il problema di assegnare quali valori possa effettivamente acquistare il genere lineare $p^{(1)}$ di una superficie algebrica di dato genere geometrico p_g non è ancora compiutamente risolto. L'Autore, proseguendo alcune sue recenti ricerche [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 276–281, 404–409; MR **20** #3865] nelle quali ha portato significativi contributi per le superficie con $p_g>3$, si occupa qui di superficie con $p_g\leq 3$, e ricorrendo a piani quadrupli abeliani dimostra l'esistenza di superfice algebriche semplici e regolari per le quali:

$$\begin{aligned} p_g &= 0, & p^{(1)} &= 3, 4, \cdots, 7, \\ p_g &= 1, & p^{(1)} &= 2, 3, \cdots, 9, 11, \cdots, 15, \\ p_g &= 2, & p^{(1)} &= 3, 4, \cdots, 17, 19, \cdots, 23, \\ p_g &= 3, & p^{(1)} &= 3, 4, \cdots, 31, \end{aligned}$$

precisando anche per quali coppie p_g, $p^{(1)}$ ($p_g\leq 3$) si hanno superficie dotate di sistema i-canonico ($i=1, 2, 3$) irriducibile oppure irriducibile e semplice. *D. Gallarati* (Genoa)

8416:

Gallarati, Dionisio. Les variétés algébriques à courbes sections canoniques. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 75–94. Centre Belge Rech. Math., Louvain, 1960.

The author discusses results of Fano [see, for instance, Pont. Acad. Sci. Comment. **11** (1947), 635–720; MR **12**, 355], of L. Roth [for instance, *Algebraic threefolds*, Springer, Berlin, 1955; MR **17**, 897], and adds his own results [Rend. Mat. e Appl. (5) **16** (1957), 315–327; MR **20** #3866] about Fano-varieties. An ample bibliography is provided.

Let S_n denote an n-dimensional projective space. Consider in S_{r+p-2} a variety M_r of dimension r and degree $2p-2$ such that the intersection of M_r with an S_{p-1} is a canonical curve of genus p.

Such varieties, called "Fano-varieties", are classified in 3 species. M_r is of the first species if every prime divisor V of M_r is linearly equivalent to mW, where W is a generic hyperplane section; of the second species if $V \equiv m\Gamma$, with $k\Gamma = W$, $k \geqq 2$. In all other cases M_r is of the third species.

Fano's M_3 of the first species exist for $p \leqq 10$. For $p \geqq 7$ they are birational. For $r > 3$ Fano's M_r of the second species exist only for $r=4$, 5 and $p=17$. Not much is known for varieties of the 3rd species.

A. Gutwirth (Haifa)

8417:

Marchionna, Ermanno. Sul teorema di Riemann-Roch relativo alle varietà algebriche. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 500–504.

This note, which continues the author's work on the Riemann-Roch theorem [same Atti (8) **24** (1958), 396–404; MR **20** #3876], introduces the irregularities q_i ($i=2, 3, \cdots, d$) of a non-singular variety V_d. In terms of these the author expresses the superabundance of a linear system $|D|$ of sufficiently general character. The main result is that the superabundance of an ample system $|X|$, which is sufficiently ample with respect to the canonical system $|K|$, is equal to $q_2+(-1)^{d+1}q_d$.

L. Roth (London)

8418:

Marchionna, Ermanno. Sul teorema di Riemann-Roch relativo alle varietà algebriche. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 672–679.

Continuing the previous note [#8417], the author proves that, if D is any irreducible non-singular hypersurface on V_d, then the adjoint system $|D'|$ cuts on D a system whose deficiency does not exceed q_d+q_{d-1}. From this result he deduces inequalities for the superabundance of $|D|$ and also of $|D'|$. He then shows that, if V_d is the complete intersection of $r-d$ forms of S_r, then V_d is both arithmetically normal and arithmetically regular (i.e., the forms of S_r of any given order cut on V_d a complete linear system of superabundance zero). It follows from this that V_d is totally regular (i.e., $q_i=0$, for all i).

A complete and simplified account of this work will be found in Appendix VI of the treatise by F. Severi: *Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica*, vol. 3, Edizoni Cremonese, Rome, 1959 [MR **21** #3409].

L. Roth (London)

8419:

Douglas, Jesse. A theorem on skew pentagons. Scripta Math. **25** (1960), 5–9.

The author considers pentagons $P_0P_1P_2P_3P_4$ of the n-dimensional euclidean space, the vertices of which are completely arbitrary. The line segments, linking the vertices with the midpoints of the opposite sides, are called the medians of the pentagon. Such is, e.g., the segment P_0M_0, where M_0 is the midpoint of the side P_2P_3. The two types of regular plane pentagon, the convex and the star regular pentagon, are considered, and pentagons which can be obtained from regular ones by affine transformation are called affine-regular. The author proves the following theorem.

Let each median P_iM_i of an arbitrary pentagon in euclidean n-space be extended through M_i to Q_i by the fraction $1/\sqrt{5}$ of its length: $M_iQ_i=(1/\sqrt{5})P_iM_i$. Then the five points Q_i in the cyclic order $i=(0, 1, 2, 3, 4)$ are the vertices of an affine-regular pentagon of convex type.

If each median P_iM_i is abbreviated to P_iQ_i' by the fraction $1/\sqrt{5}$ of its length, then the five points Q_i' are the vertices of an affine-regular pentagon of star-shaped type.

L. Gyarmathi (Debrecen)

8420:

Le Cocq, R. Faisceaux linéaires de cubiques planes. Mathesis **69** (1960), 131–135.

8421:

Hua, Lo-gèn [Hua, Loo-Keng]; Rosenfel'd, B. A. Geometry of rectangular matrices and their application to real projective and non-euclidean geometry. Sci. Sinica **6** (1957), 995–1011. (Russian)

An m-plane E_m in n-dimensional projective space P_n is determined by $m+1$ points, or by the $(m+1)\times(n+1)$ matrix Ξ of the coordinates of these points. Denoting the matrix of the first $m+1$ rows of Ξ by X_0, and the matrix consisting of the other $n-m$ rows by X_1, E_m is determined by the pair (X_0, X_1). (Y_0, Y_1) determines the same E_m if and only if $Y_0=X_0K$, $Y_1=X_1K$ for some non-singular matrix K. If X_0^{-1} exists, $X=X_1X_0^{-1}$ is called the affine matrix coordinate of E_m, and is unique. This situation is in formal analogy with the case of points in ordinary projective space: the m-planes can be regarded as the points of $(n-m)/(m+1)$-dimensional projective space over the coefficient domain of $(m+1)\times(m+1)$ matrices. In the present paper this analogy is investigated and used to give a simple approach to some geometric results about the Grassmanian. The effect of a collineation on the affine matrix coordinate of m-planes is a transformation $X\to(C+DX)(A+BX)^{-1}$, where A, B, C, D are matrices. Duality has a natural interpretation. The cross-ratio of four hyperplanes is defined analogously to the ordinary cross-ratio, and is a square matrix determined up to similarity (just like in the case of the Siegel half-space). The eigenvalues of this matrix turn out to be identical with certain invariants introduced earlier by B. Segre. Applications to non-Euclidean geometry are also indicated.

A. Korányi (Berkeley, Calif.)

8422:

Hua, Lo-gèn; Rozenfel'd, B. A. Geometry of rectangular matrices and its application to real projective and non-euclidean geometry. Acta Math. Sinica **8** (1958), 132–145. (Chinese. Russian summary)

Translation of #8421 reviewed above.

8423:

Rozenfel'd, B. A. Rectangular matrices and non-euclidean geometries. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 21–48. (Russian)

Somewhat more detailed and more systematic exposition of the material contained in the paper reviewed above [#8421].

A. Korányi (Berkeley, Calif.)

8424:

Ahrens, Joachim. Eine Kennzeichnung der orthogonalen Gruppen vom Index Null. Arch. Math. **11** (1960), 116–126.

Sei G eine orthogonale Gruppe $O_n(K,f)$ $(n\geqq 3)$ vom Index 0 über einem Körper K von Charakteristik $\neq 2$; die Symmetrien (Hyperebenen-Spiegelungen) $\alpha, \beta, \cdots$ aus G sind involutorisch und bilden ein Erzeugendensystem S von G; es gelten die vier folgenden Gesetze (wir schreiben $\alpha|\beta$ für: $\alpha\beta$ ist involutorisch, und $\alpha_1|\alpha_2|\cdots|\alpha_k$ für: $\alpha_i|\alpha_j$ für $i\neq j$): V. Zu $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ gibt es ein ε mit $\varepsilon|\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$. T. Aus $\alpha\neq\beta$ und $\alpha\beta\gamma$, $\alpha\beta\delta\in S$ folgt $\alpha\gamma\delta\in S$. N. Gilt $\alpha_1|\alpha_2|\cdots|\alpha_n$ und $\beta_1|\beta_2|\cdots|\beta_n$, so ist $\alpha_1\alpha_2\cdots\alpha_n=\beta_1\beta_2\cdots\beta_n\neq 1$. Z. Zu α, β mit $\alpha|\beta$ gibt es ein ε, so dass $\varepsilon\alpha\neq\alpha\varepsilon$ und $\varepsilon\beta\neq\beta\varepsilon$ ist.

Verf. zeigt, dass diese vier Gesetze bereits die genannten orthogonalen Gruppen charakterisieren: Sind eine abstrakte Gruppe G und ein aus involutorischen Elementen $\alpha, \beta, \cdots$ bestehendes Erzeugendensystem S von G sowie eine natürliche Zahl $n\geqq 3$ gegeben und gelten für sie V, T, N, Z, so gibt es einen Isomorphismus von G auf eine orthogonale Gruppe der genannten Art, welcher S auf die Gesamtheit ihrer Symmetrien abbildet.

Zum Beweis leitet Verf. zunächst aus den Axiomen V, T, N, Z einige Sätze her, die von ähnlicher Art wie die Axiome sind, insbesondere den "Satz von den k Spiegelungen" und seine Umkehrung, die es gestatten, die Verkürzbarkeit von Produkten von Erzeugenden zu beherrschen. Dann wird unter Benutzung der verbandstheoretischen Charakterisierung der projektiven Räume ein projektiver Raum P_{n-1} konstruiert, dessen Punkte die Elemente von S sind; als Teilräume werden die Punktmengen $M^\perp=\{\alpha:\alpha|\beta \text{ für alle } \beta\in M\}$ genommen, wobei M eine beliebige Punktmenge ist. In P_{n-1} ist $T\to T^\perp$ eine Polarität ohne selbstkonjugierte Elemente. Die Ebenen von P_{n-1} sind elliptische Ebenen; hieraus wird geschlossen, dass der Koordinaten-Schiefkörper K von P_{n-1} kommutativ und von Charakteristik $\neq 2$ und dass die Polarität $T\to T^\perp$ linear ist. Die Polarität wird daher durch eine symmetrische Bilinearform f vom Rang n und Index 0 vermittelt. P_{n-1} lässt sich somit als Teilraumverband des metrischen Vektorraumes $V_n(K,f)$ auffassen. Die Gruppe der inneren Automorphismen von G lässt sich als die projektiv-orthogonale Gruppe $PO_n(K,f)$ auffassen, und schliesslich wird der hiermit gegebene Isomorphismus von G/Z (Z das Zentrum von G) auf $PO_n(K,f)$ zu einem Isomorphismus von G auf $O_n(K,f)$ verfeinert.

F. Bachmann (Kiel)

8425:

Joussen, Jakob. Ordnungsfunktionen in freien Ebenen. Abh. Math. Sem. Univ. Hamburg **24** (1960), 239–263.

Die von E. Sperner begonnenen [Math. Ann. **121** (1949), 107–130; S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1949**, no. 10, 413–448; Abh. Math. Sem. Univ. Hamburg **16** (1949), nos. 3–4, 140–154; MR **11**, 197; **12**, 43, 197] und von H. Karzel weitergeführten [Math. Ann. **127** (1954), 228–242; Math. Z. **62** (1955), 268–291; **64** (1956), 131–137; MR **15**, 818; **17**, 293, 1123] Untersuchungen über den Zusammenhang zwischen geometrischer und algebraischer Anordnung, die die Frage nach einer Ordnungsfunktion in nichtdesarguesschen projektiven Ebenen, die eine nichttriviale Halbordnung des Koordinatenbereiches liefert, noch offen ließen, führt Verf. hier auf rein geometrischem Wege weiter für die Hall'schen freien Ebenen F_m [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72]. Er zeigt, daß in einer F_m Ordnungsfunktionen existieren, die die Geradenrelation erfüllen; er gewinnt auf konstruktivem Wege eine Übersicht über die Gesamtheit dieser Ordnungsfunktionen. Es gilt der Satz, daß für eine halbprojektive reguläre Inzidenzstruktur J sich jede definite Ordnungsfunktion von J fortsetzen läßt zu einer definiten Ordnungsfunktion der ersten Oberstruktur von J. Eine Ordnungsfunktion heißt definit, wenn sie der sogenannten charakteristischen Bedingung genügt, eine Bedingung, die in der projektiv abgeschlossenen Ebene nachträglich mit der Geradenrelation äquivalent ist und in einer halbprojektiven Inzidenzstruktur, die wenigstens ein Viereck enthält, notwendig und hinreichend dafür ist, daß eine Halbordnung die Geraden- und die Vierecksrelation erfüllt.—Für die die Geradenrelation erfüllenden Ordnungsfunktionen von F_m wird erstmalig eine Basisdarstellung B angegeben; sie ist für eine Inzidenzstruktur J allgemein definiert als Teilmenge der nichtinzidenten Paare Punkt, Gerade aus J mit der Eigenschaft, daß jede definite Ordnungsfunktion in J durch ihre Werte auf B eindeutig bestimmt ist und daß jede Belegung von B mit Werten $+1$ oder -1 sich ergänzen läßt zu einer definiten Ordnungsfunktion von J.—Die auf Sperner zurückgehende Einteilung der Ordnungsfunktionen in Äquivalenzklassen führt abschließend zu einer vollständigen Übersicht über alle Klassen definiter Ordnungsfunktionen einer freien Ebene und ihre einfache Beschreibung. Die Anzahl dieser Klassen ist endlich.

R. Moufang (Frankfurt a.M.)

8426:

Lüneburg, Heinz. Über die beiden kleinen Sätze von Pappos. Arch. Math. **11** (1960), 339–341.

It is shown that a finite projective plane is desarguesian if (and obviously only if) the little Reidemeister condition and a certain partial Thomsen condition hold on it; hence, on a finite plane, the general Pappus theorem follows from either of the two little (axial or central) Pappus theorems, so that each of the latter theorems implies the other one (to decide whether similar results hold on infinite planes are still open questions). *B. Segre* (Rome)

8427:

Lombardo-Radice, L. Gruppi di collineazioni dei piani grafici finiti. Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 114–136. Edizioni Cremonese, Rome, 1960.

As yet, a theory of collineations is very far from being completed in the case of finite non-(micro)desarguesian planes. The present paper gives a survey of the results on the subject known up to the beginning of 1960, with the addition of a few interesting remarks and open questions; it deals especially with planes on Hall's or on associative quasicorpora of order >9, with planes on certain distributive quasicorpora, with Hughes' and with André's planes. *B. Segre* (Rome)

8428:

Bruck, R. H. Quadratic extensions of cyclic planes. Proc. Sympos. Appl. Math., Vol. 10, pp. 15–44. American Mathematical Society, Providence, R.I., 1960.

On sait que tout plan projectif arguésien fini est

cyclique; mais il existe des plans cycliques infinis qui ne sont pas arguésiens et l'on ignore si l'on peut construire des plans cycliques finis non arguésiens. T. A. Evans et H. B. Mann [Sankhyā **11** (1951), 357–364; MR **13**, 899] ont montré que tout plan projectif cyclique fini d'ordre inférieur à 1600 est d'ordre p^k (p premier, $k \in N^+$). M. Hall [Proc. Amer. Math. Soc. **7** (1956), 975–986; MR **18**, 560] a étudié l'isomorphisme des plans projectifs cycliques d'ordre p^k jusqu'à $p=7$, $k=2$. Après une longue série de lemmes concernant l'extension quadratique d'un ensemble différence d'ordre m ($m \geqq 2$) à un ensemble différence d'ordre m^2, on parvient à deux théorèmes affirmant l'équivalence de tout ensemble différence D_{49} [resp D_{81}] avec un ensemble différence construit à partir de D_7 [resp D_9]. Utilisant la correspondance entre les plans cycliques et les ensembles différence, il suit de là que, pour les valeurs m de 2 à 10, le plan projectif cyclique d'ordre $n=m^2$ est unique et par conséquent arguésien. Le résultat, déjà connu pour $m=2$ à 6, est nouveau pour $m=7$ à 10. Une proposition obtenue incidemment est que tout plan projectif cyclique π, d'ordre $n=m^2$ ($m \geqq 2$), peut être décomposé en $m^2-m+1=s$ sous-plans projectifs $\pi_1, \pi_2, \cdots, \pi_s$, tous d'ordre m et tels que chaque point et chaque droite de π appartienne à un π_i et à un seul.

A. Sade (Marseille)

8429:

Segre, Beniamino. Le geometrie di Galois. Archi ed ovali; calotte ed ovaloidi. Confer. Sem. Mat. Univ. Bari **43–44**, 31 pp. (1959).

This is an expository paper, with proofs either omitted or very briefly sketched. The author collects recent results —obtained mostly by Italian mathematicians, and primarily by the author himself—on the following subject: Let $S_{r,q}$ be the r-dimensional projective space over GF(q). Then a set of K points in $S_{r,q}$ is termed a K-calotte denoted by $K^*_{r,q}$ if no three points of $K^*_{r,q}$ are collinear. $K^*_{r,q}$ is called complete if it is not contained in a $(K+1)^*_{r,q}$, and it is an ovaloid if there is no $L^*_{r,q}$ with $L>K$. For $r=2$, the author speaks of k-arcs and ovals rather than K-calottes and ovaloids. The general problem is to classify, for all r, q, all complete K-calottes (or k-arcs) and in particular all ovaloids (ovals). The author's theorem that if $r=2$ and q is odd every oval is a conic [cf. Canad. J. Math. **7** (1955), 414–416; MR **17**, 72] can be regarded as the starting point for this theory, and the paper under review demonstrates impressively how rapid progress has been in this field during the last five years; the valuable bibliography contains forty-six entries. The paper contains a wealth of results which cannot all be reproduced here. After a discussion of conics for ($r=2$) and quadrics ($r>2$) the author turns to a thorough discussion of the plane case $r=2$. He gives necessary conditions for the existence of complete k-arcs in arbitrary (not necessarily desarguesian) finite projective planes which lead to a number of nonexistence theorems for such arcs. After this he turns to the deeper results of the theory which generalize the theorem mentioned above: Let $t>0$ be an integer; if q is even and $\geqq t^2-t-[(t-1)/2]$, then a $(q-t+2)$-arc in $S_{2,q}$ is incomplete but is contained in a unique complete $(q+2)$-arc. If q is odd and $\geqq 16t^2-t-37$, then every $(q-t+2)$-arc in $S_{2,q}$ is contained in a unique nondegenerate conic which is a complete $(q+1)$-arc. A nondegenerate conic Q is a $(q+1)$-arc for arbitrary q, but for even q it is not complete: all its tangents pass through the same point $P \notin Q$, and P and Q together form an oval in $S_{2,q}$. If $q=2^h$, $1 \leqq h \leqq 3$ (and perhaps $h=6$, this case being undecided), every oval in $S_{2,q}$ is of this type, but for all other h there are ovals which cannot be obtained from conics in the manner described. The author gives a method of constructing such ovals for $h=5$ and $h \geqq 7$; for $h=4$ a counterexample has been found by an electronic computer. The situation is similar in the case $r=3$. If q is odd and $K \geqq q^2-q+19$, every $K^*_{3,q}$ is contained in an elliptic quadric which is a complete (q^2+1)-calotte in $S_{3,q}$. In particular, for odd q every ovaloid in $S_{3,q}$ is an elliptic quadric. {Reviewer's remark: If q is even, there are (q^2+1)-calottes in $S_{3,q}$ which are not quadrics. Such calottes have recently been used by Tits for a representation of Suzuki's new simple groups.} Little seems to be known about the classification problem for $r \geqq 4$. The author calls $[M]^*_{r,q}$ the number of points of any ovaloid in $S_{r,q}$ and proves some inequalities for this function $[M]^*_{r,q}$, $r \geqq 4$. The paper concludes with a discussion of certain types of algebraic varieties in $S_{r,q}$ which can be regarded as generalizations of calottes.

P. Dembowski (Frankfurt a.M.)

8430:

Wong, Yung-Chow Clifford parallels in elliptic $(2n-1)$-space and isoclinic n-planes in Euclidean $2n$-space. Bull. Amer. Math. Soc. **66** (1960), 289–293.

In an elliptic space El^{2n-1} of dimension $2n-1$ (that is, a projective space turned metric by a non-degenerate imaginary hyperquadric) two $(n-1)$-planes are said to be Clifford-parallel if the distance from the second plane to any point of the first plane is the same. This parallelism is reflexive and symmetric, but not transitive. A set of $(n-1)$-planes in El^{2n-1} is a maximal set if every $(n-1)$-plane in the set is Clifford-parallel to every other one and if the set is not a subset of a larger one with the same property. Existence of maximal sets is established and it is shown that a maximal set is, in a certain sense, a linear set. The proofs lead to the system of matrix equations

$$B_h+B_h' = 0, \quad B_h^2 = -1, \quad B_hB_k+B_kB_h = 0,$$

which first appeared in Hurwitz' theorem on the composition of quadratic forms. *H. Schwerdtfeger* (Montreal)

CONVEX SETS AND GEOMETRIC INEQUALITIES

8431:

Fejes Tóth, L. Neuere Ergebnisse in der diskreten Geometrie. Elem. Math. **15** (1960), 25–36.

This is a survey of the kind of problems discussed in the author's *Lagerungen in der Ebene, auf der Kugel und im Raum* [Springer, Berlin, 1953; MR **15**, 248], brought up to date by the inclusion of more recent work of Bambah, Besicovitch and Eggleston, Coxeter, Danzer, Davenport, Erdös, Few, Florian, Grünbaum, Heppes, Lenz, Molnár, Rankin, Rogers, Schütte, Sperling, Steinhaus and G. L. Watson, as well as by the author himself. For instance, there is the problem of filling the surface of a sphere with n equal non-overlapping circles, as large as possible. For $n=12$, the circles are inscribed in the faces of a regular dodecahedron. L. Danzer has found that, for $n=11$, eleven of these twelve circles still give the best

arrangement. This 'problem of Tammes' has now been solved for every $n \leq 12$ except $n=10$. Another example is the theorem of Besicovitch and Eggleston [Quart. J. Math. Oxford Ser. (2) **8** (1957), 172–190; MR **20** #1950]: of all polyhedra with a given insphere, the cube has the smallest total length of edges. The author raises the corresponding problem with the restriction to triangular faces. Observing that the regular tetrahedron and octahedron, with a unit insphere, each have total edge length $6^{3/2}$, he conjectures that the problem has these two solutions, i.e., that every other triangular polyhedron has a greater total length of edges. *H. S. M. Coxeter* (Toronto)

8432:

Fejes Tóth, L. Über eine Volumenabschätzung für Polyeder. Monatsh. Math. **64** (1960), 374–377.

Let f and k denote the numbers of faces and edges of a convex polyhedron contained in the unit sphere. The author proves that the volume V of the polyhedron satisfies the inequality $V \leqq S$, where

$$S = \max\{n_1 U(\tau_1, p_1) + n_2 U(\tau_2, p_2)\},$$

n_1, n_2, p_1, p_2 are integers defined by the relations $n_1+n_2=f$, $n_1p_1+n_2p_2=2k$, $p_1-p_2=1$, the maximum extends over the values of τ_1 and τ_2 given by the inequalities $\tau_1 \geqq 0$, $\tau_2 \geqq 0$, $n_1\tau_1+n_2\tau_2 \leqq 4\pi$, and

$$U(\tau, p) = \frac{p}{3}\cos^2\frac{\pi}{p}\tan\frac{2\pi-\tau}{2p}\left(1-\cot^2\frac{\pi}{p}\tan^2\frac{2\pi-\tau}{2p}\right).$$

(There is a misprint: k for $2k$.)

Observing that V attains its bound S for each of the Platonic solids, the author compares the values of V and S (to three decimal places) for other uniform polyhedra inscribed in the unit sphere; in particular: for the cuboctahedron, $V=2.357$, $S=2.399$; for the snub cube, $V=3.251$, $S=3.269$; for the rhombicuboctahedron, $V=3.183$, $S=3.195$; and for the truncated icosahedron, $V=3.633$, $S=3.642$. Since the difference is least (both absolutely and relatively) in the last case, and since no polyhedron with 32 faces can have more than 60 vertices, it is a natural conjecture that, among all convex polyhedra with at most 32 faces, contained in a fixed sphere, the truncated icosahedron has the greatest possible volume. *H. S. M. Coxeter* (Toronto)

8433:

Heppes, A.; Szüsz, P. Bemerkung zu einer Arbeit von L. Fejes Tóth. Elem. Math. **15** (1960), 134–136.

The authors consider the problem of dissecting a disk of variable density into a great number of pieces such that the sum of the moments of inertia of the pieces, about their respective centroids, is as small as possible. It was shown by Fejes Tóth [Acta Math. Sci. Hungar. **10** (1959), 299–304; MR **22** #4016] that the pieces are "nearly regular" hexagons whose moments of inertia are all equal. The authors prove that, if the curves of constant density are closed, the dissection will always contain a "line of dislocation" (in the crystallographic sense). They give a very remarkable diagram to illustrate the case when the density is inversely proportional to the distance from the center. *H. S. M. Coxeter* (Toronto)

8434:

Haupt, Otto. Über einige Grundeigenschaften der Bogen ohne $(n-2, k)$-Sekanten im projektiven P_n $(n \leqq k)$. Math. Ann. **139**, 151–170 (1959).

This is the first of a series of papers which intend to apply direct geometric concepts and methods to Barner's n-dimensional generalization of the Four-vertex Theorem [Abh. Math. Sem. Univ. Hamburg **20** (1956), 196–215; MR **19**, 60]. A curve [arc] B in projective n-space is the homeomorphic image of a circle [segment]. The multiplicity with which an r-space meets B is defined without differentiability assumptions. B is "free from n-secants" if no $(n-2)$-space meets B more than $(n-1)$ times. B is "strictly free from n-secants" if in addition every $(n-2)$-space which meets B $(n-1)$ times lies in some $(n-1)$-space which meets B only with the same multiplicity [Barner's "strictly convex" arcs are strictly free from n-secants]. Some basic properties of B's which are free or strictly free from n-secants are collected. The following topics may be mentioned: support and intersection, paratingents, central projection. The precise formulations are too long for quoting. *P. Scherk* (Toronto)

8435:

Gale, David; Klee, Victor. Continuous convex sets. Math. Scand. **7** (1959), 379–391.

If E is a finite-dimensional Euclidean space with inner product $(\ ,\ )$ and if X is a subset of E, then the support function σ for X is defined by $\sigma(u)=\sup\{(x, u) : x \in X\}$ for each unit vector u. The set X is called continuous if its support function is continuous on the unit sphere. An asymptote of X is a ray in the complement of X at zero distance from X. Theorem 1.3 asserts (among other things) the equivalence, for X closed and convex, of the following: (i) X is continuous; (ii) X has no asymptote and contains no ray in its boundary; (iii) for each p in X, the convex hull of X and p is closed; (iv) each closed convex $Y \subset E \sim X$ is a positive distance from X; (v) if Y is closed and convex then so is $X+Y$. These are familiar properties of the class of all compact convex sets; the continuous convex sets form (in a reasonable sense) a maximal class with these properties. Several theorems are proved concerning the closed convex hull of the union of an arbitrary family of continuous convex sets; an easy corollary states that the convex hull of a finite union of such sets is closed. Some generalized separation theorems are also proved. For instance, Theorem 2.8 implies that if $\{X_s : s \in S\}$ is a family of continuous proper subsets of E having empty intersection, then there exist closed halfspaces $Q_s \supset X_s$ such that $\bigcap Q_s$ is empty. *R. R. Phelps* (Berkeley, Calif.)

8436:

Eggleston, H. G. Squares contained in plane convex sets. Quart. J. Math. Oxford Ser. (2) **10** (1959), 261–269.

The following two results are proved: (a) the minimum width of a convex set that contains a given square is largest when the convex set is a regular triangle; (b) the width of a set of constant width that contains a given square is largest when the set of constant width is a Reuleaux triangle. If s denotes the side-length of the square and Δ the minimum width in (a) and the width

in (b), then in (a) $s \geqq 2(2-\sqrt{3})\Delta$ and in (b) $s \geqq 2k$, where k is the positive root of

$$4k^2+(1+2\sqrt{3})k\Delta+\Delta^2-(4k+\sqrt{3}\Delta)(\Delta^2-k^2)^{1/2} = 0.$$

Approximately, k is 0.3237Δ.

P. C. Hammer (Madison, Wis.)

DIFFERENTIAL GEOMETRY

See also 8077, 8434, 8522, 8523, 8526, 8527, 8529, 8530, B8569, B9231, B9243, B9244.

8437:

Bakel'man, I. Ya. Differential geometry of smooth manifolds. Amer. Math. Soc. Transl. (2) **16** (1960), 357–358.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 145–146 [MR **19**, 165].

8438:

Schaal, Hermann. Über Regelflächen mit ähnlichen Lieschen Schmiegflächen 2. Ordnung und die Spitzen der scheinbaren Umrisse von Regelschraubenflächen bei Parallelprojektion. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 111–182.

Étant donnée une surface réglée gauche Ψ, soit F^2 sa quadrique osculatrice de Lie correspondant à une de ses génératrices e_1. L'auteur considère d'abord les surfaces réglées Φ pour lesquelles toutes les F^2 sont semblables à une surface réglée donnée F_0 du second ordre. Il envisage ensuite deux cas où les génératrices de contact des différentes F^2 correspondent soit à des génératrices différentes, soit à une même génératrice de F_0. En utilisant la propriété d'après laquelle toutes les F^2 de Φ sont semblables à F_0 donnée, il est possible de donner une construction commune concernant les surfaces Φ. En particulier, la détermination par l'emploi de la règle et du compas des points de rebroussement du contour apparent des hélicoïdes réglés projetés parallèlement devient possible dans certains cas.

F. Şemin (Istanbul)

8439:

Wunderlich, W. Geometrische Betrachtungen um eine Apfelschale. Elem. Math. **15** (1960), 60–66.

A propos du pelage d'une pomme, l'auteur étudie d'abord géométriquement, ensuite analytiquement, ce qui suit: Étant donné un cône de révolution Δ de sommet O, soient m, $\overline{m}$ les cercles d'intersection de Δ avec la sphère $\Pi(O, R)$ de centre O et de rayon R. P étant un point quelconque du grand cercle d'intersection C de Π avec un plan tangent à Δ, si l'on fait tourner C autour de son axe tandis que son plan roule autour de Δ en lui restant constamment tangent, le point P décrit sur Π une courbe d'égale pente k qui peut être considérée comme une développante sphérique de m et de $\overline{m}$. Après un tour complet, un point P_0 de k occupe une nouvelle position P_1. Le segment de droite P_0P_1 engendre une portion de surface de développable Σ dont l'arête de rebroussement f est une géodésique de Δ. Le lieu 1 du milieu Q_0 de P_0P_1 est également une courbe du même genre que k, mais située sur la sphère $\theta(O, r)$. On envisage finalement l'applicabilité de Σ sur un plan.

F. Şemin (Istanbul)

8440:

Nitsche, Johannes C. C. Ein Eindeutigkeitssatz für zweifach zusammenhängende Minimalflächen. Math. Z. **74** (1960), 289–292.

In an earlier work [J. Math. Mech. **6** (1957), 859–864; MR **20** #6115] the author has proved that if a minimal surface has for all z a representation in cylindrical coordinates of the form $r=r(z, \varphi)$ with $r(z, \varphi+2\pi)=r(z, \varphi)$, and if all sections $z=$const of the surface are convex curves, then the surface is necessarily a catenoid. In the present note the assumption of convexity is shown to be superfluous. The proof is obtained in a simple and elegant way from the fact that the angle made with the positively directed x-axis by a tangent to the section $z=$const is harmonic on a minimal surface. {The reviewer remarks that a more perspicuous proof of this fact than the one indicated by the author can be obtained from the observations that this angle differs by $\frac{1}{2}\pi$ (mod π) from the argument of the stereographic projection from the north pole of the spherical image of the surface, and that this mapping is conformal on the surface.}

R. Finn (Stanford, Calif.)

8441:

Backes, F. Une cyclide attachée en tout point d'une surface minima. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 132–139.

By considering the isotropic congruence associated with a minimal surface, the author shows how a cyclide can be associated with each of its points.

T. J. Willmore (Liverpool)

8442:

Drăgilă, Pavel. Sur certaines surfaces affinement identiques. Boll. Un. Mat. Ital. (3) **15** (1960), 162–169. (Italian and English summaries)

In this note the author continues the study of pairs of surfaces which are affinely equivalent in a sense defined in a previous note [same Boll. (3) **13** (1958), 465–469; MR **21** #2253]. The surfaces in question have the property that an asymptotic tangent to the first surface is parallel to the asymptotic tangent at the corresponding point of the second. Having obtained the equations of such surfaces in the note cited, he investigates the possibility of the existence of other surfaces than those already found. The main result of the paper is that no such surfaces exist. In the second part of the note he obtains some further properties of the surfaces already found. He reproduces the definition of a type of correspondence which he calls "transport parallèle distancié" and proves that the tangents at corresponding points of two affinely equivalent surfaces correspond in this way.

E. T. Davies (Southampton)

8443:

Cossu, A. Sulle calotte del second'ordine corrispondenti in una trasformazione puntuale tra due spazi proiettivi. Ann. Mat. Pura Appl. (4) **47** (1959), 151–167.

Let S_3, $\bar{S}_3$ be two projective spaces; T a local mapping, point $O \in S_3$, point $\bar{O} \in \bar{S}_3$, $\bar{O}=TO$, with the jacobian of T at O, $\bar{O}$ different from zero. Let Ω be one of the ∞^3

collineations equivalent to T in the neighborhood of the first order of O. Consider a surface element of the 2nd order, or cap, σ with center at O; let $\bar{\sigma}=T\sigma$, $T^*=\Omega^{-1}T$. It is shown: (1) the set of all caps with center at O and the same tangent plane π is a vector space; (2) if ω is the inflexional cap with center O belonging to π, and σ tangent to π at O, then $\sigma^*=T^*\sigma=\sigma+T^*\omega$; $\bar{\sigma}=\Omega\sigma+T\omega$. Further information on the behavior of T^* is given by the correspondence obtained by associating to a line through O (i.e., its inflexional element, its transformed element by T^* and its plane) a plane through the line. This allows a thorough study of the caps which are transformed by T in parabolic caps. *E. Bompiani* (Rome)

8444:

Speranza, Francesco. Le V_5 che posseggono ∞^{11} E_2 di $\gamma_{1,3}$. Boll. Un. Mat. Ital. (3) **14** (1959), 206–213. (English summary)

È noto che se una V_5 possiede ∞^3 E_2 appartenenti a quasi-asintotiche $\gamma_{1,3}$ risulta $\delta\leq 13$, e gli S_5 tangenti di V_5 invadono una $W_{19-\delta}$. Proseguendo le ricerche di M. Villa relative ai casi $\delta=13$ e $\delta=12$, l'A. determina qui le V_5 con $\delta=11$. Esse sono: (a) gli S_1-coni proiettanti V_3 dotate di ∞^4 $\gamma_{1,3}$; (b) le V_5 di S_8 i cui spazi tangenti invadono l'S_8; (c) le ∞^2 di V_3 negli S_4 di un S_3-cono proiettante una superficie di Veronese; (d) il cono proiettante da un punto una $V_4{}^6$ di Corrado Segre. *D. Gallarati* (Genoa)

8445:

Backes, F. Sur les surfaces $\mathscr{P}$ de la géométrie différentielle projective. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 345–354.

In a previous paper the author has defined $\mathscr{P}$-surfaces [same Bull. (5) **44** (1958), 457–465; MR **20** #5954]. Here he obtains relations between $\mathscr{P}$-surfaces and the R-surfaces considered by A. Demoulin [C. R. Acad. Sci. Paris **153** (1911), 590–593]. *T. J. Willmore* (Liverpool)

8446:

Degen, Wendelin. Über geschlossene Laplace-Ketten ungerader Periodenzahl. Math. Z. **73** (1960), 95–120.

The parameter curves $u=\text{const}$ and $v=\text{const}$ of a two-dimensional surface $\mathfrak{x}=\mathfrak{x}(u,v)$ in projective R_n form a conjugate net if the points $\mathfrak{x}$, $\mathfrak{x}_u$, $\mathfrak{x}_v$, $\mathfrak{x}_{uv}$ are coplanar. Their tangents then envelop two surfaces $\mathfrak{x}_1$ and $\mathfrak{x}_{-1}$ whose parameter curves again form conjugate nets. Repeating this procedure as often as possible we obtain a Laplace chain $\cdots$, $\mathfrak{x}_{-2}$, $\mathfrak{x}_{-1}$, $\mathfrak{x}_0=\mathfrak{x}$, $\mathfrak{x}_1$, $\mathfrak{x}_2$, $\cdots$. Assume that $n=2r+1$ is odd, that $\mathfrak{x}_{n+2}=\mathfrak{x}_0$ and that the points $\mathfrak{x}_i=\mathfrak{x}_i(u,v)$ span R_n for each u, v. Any n consecutive points $\mathfrak{x}_{i-r}$, $\cdots$, $\mathfrak{x}_i$, $\cdots$, $\mathfrak{x}_{i+r}$ then span a hyperplane $\mathfrak{X}_i$. The $\mathfrak{x}_i$ and $\mathfrak{X}_i$ can be normed such that

$$\sum_0^{n+1}\mathfrak{x}_i=\sum_0^{n+1}\mathfrak{X}_i=0;$$

$$\begin{aligned}\mathfrak{x}_i\mathfrak{X}_k &= 1 \text{ if } k=i+r+1,\\ &= -1 \text{ if } k=i-r-1,\\ &= 0 \text{ otherwise}\end{aligned}$$

$$(i,\,k=0,\,1,\,\cdots,\,n+1).$$

For each u, v the correlation which maps each $\mathfrak{x}_i$ onto $\mathfrak{X}_i$ is a null-system. If the family of these null-systems is one-dimensional, the Laplace chain is called an R-chain. The R-chains are characterized by the fact that the congruences $(\mathfrak{x}_i, \mathfrak{x}_{i+1})$ and $(\mathfrak{X}_i, \mathfrak{X}_{i+1})$ have the same Darboux invariants; $i=0,\,1,\,\cdots,\,n+1$. The parameters u, v can then be replaced by parameters s, t with the following properties: The null-systems depend only on s. Put $\mathfrak{p}(s,\,t)=\sum p_i\mathfrak{x}_i(s,\,t)$, $\mathfrak{P}(s,\,t)=\sum p_i\mathfrak{X}_i(s,\,t)$. Then $\mathfrak{p}\mathfrak{P}=(\partial\mathfrak{p}/\partial s)\mathfrak{P}=\mathfrak{p}(\partial\mathfrak{P}/\partial s)=0$. The curves $\mathfrak{p}(s,\,\text{const})$ lie in r-spaces. The mappings $\mathfrak{p}(s_0,t)\rightarrow\mathfrak{p}(s,t)$ are projectivities independent of t. They form a group which maps the R-chain onto itself, permuting the t-curves. *P. Scherk* (Toronto)

8447:

Trušin, G. I. Third order conjugate systems and the problem of their focal transformations. Dokl. Akad. Nauk SSSR **131** (1960), 265–268 (Russian); translated as Soviet Math. Dokl. **1**, 245–248.

Such systems, which depend on twelve functions of three arguments, are investigated by a reference system in which the basic points A, A_{α_1} form the basis for the first osculating space (tangent space), α_1, i, j, $k=1$, $\cdots$, p; the points A, A_{α_1}, A_{α_2} the basis for the second osculating space, etc. The forms ϕ^{α_2}, ϕ^{α_3} are reduced to canonical (one term) forms. Forms like (ij) are used for α_2. Then, taking $p=3$, a first holonomic conjugate system is defined if there exists on every three-dimensional space $(AA_i,\,A_j,\,A_{(ij)})$ at least one point P which is not moved outside the four-dimensional space $(AA_1A_2A_3A_{(ij)})$, when A is displaced in direction A_k. A second holonomic system is defined if there exists on every one of the four-dimensional surfaces $(AA_1A_2A_3A_{(ij)})$ at least one point P which does not move outside this space when A is displaced in direction AA_k. It is stated that both systems exist with a degree of freedom expressed by six functions of three arguments, and are stratified into one-parametric families of two-dimensional surfaces through conjugate lines of the set. Then principal holonomic conjugate systems are introduced, which are three-dimensional surfaces with an invariant net of basic lines ω^1, ω^2, ω^3 on which out of every complex of lines $AA_{(ij)}$ by motion in direction AA_k a developable surface is generated. Such principal systems exist and depend on three functions of three arguments. *D. J. Struik* (Cambridge, Mass.)

8448:

Godeaux, Lucien. Recherches sur la théorie des congruences de droites. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 299–309.

The goal of this paper is to study certain congruences of lines with the help of the representation on the Klein quadric Q in S_5. W-congruences are not considered and neither are congruences whose focal surfaces are ruled. The notation and results of the author's book *La théorie des surfaces et l'espace réglé* [Hermann, Paris, 1934] are used.

First the author proves two theorems of Koenigs which consider the ruled surfaces of a congruence passing through a particular line j of the congruence and the semi-quadrics osculating these ruled surfaces along j. Then, letting X, Y represent on Q the tangents to the asymptotics ξ, η at a point x of one of the focal nappes, he computes the first two Laplace transforms of X in the sense of η and the first two of Y in the sense of ξ. The computations are to serve as an introduction to a study

(in subsequent papers) of the relations between the Lie quadrics of the focal nappes. *A. Schwartz* (New York)

8449:

Kovancov, N. I. Canonical quadrics of a complex of lines in projective space. Mat. Sb. (N.S.) **50** (**92**) (1960), 129–170. (Russian)

An analogy is drawn between the problem of assigning a projective normal to a surface in projective three-space, and that of finding an exclusive pair of points on a ray of a line complex. The points of such a pair, once found, can be taken as vertices of a coordinate tetrahedron. The principal quadrics of the complex can serve as analogous to the asymptotic lines on a surface, and the analogy can be expressed in terms of Pfaffians. This leads to a detailed study of the canonical tetrahedra described by the author in Ukrain. Mat. Ž. 8 (1956), 140–158 [MR 18, 505], which are invariantly connected with two other tetrahedra obtained from the first by certain linear transformations. Among the generators of a principal quadric to which a ray of the complex belongs, a certain generator l can be singled out by a construction involving elements of the third order of the neighborhood of the ray. This line l, together with the two lines corresponding to it in the linear transformations of the tetrahedra, determines a canonical quadric, which lies symmetrically with respect to the triplet of principal surfaces to which a generating line of the complex belongs. A detailed investigation of these canonical quadrics follows.

D. J. Struik (Cambridge, Mass.)

8450:

Švec, Alois. Quelques remarques au sujet de la théorie des surfaces réglées dans des espaces projectifs de dimension impaire. Czechoslovak Math. J. **10** (**85**) (1960), 309–315. (Russian summary)

E. Čech a introduit, pour l'étude des surfaces réglées dans les espaces projectifs de dimension impaire $2n+1$, des formes quasiflecnodales $f_i(t)$ au nombre de n ainsi que n invariants j_i. Dans S_3, la condition pour que deux surfaces réglées π, $\bar{\pi}$ soient déformées projectives l'une de l'autre est que leurs formes flecnodales soient égales. En utilisant la notion de droite totalement K-linéarisante pour une correspondance T entre deux surfaces réglées de S_{2n+1}, l'auteur détermine les caractères géométriques de la correspondance $T_{pq}{}^0$ pour laquelle on réalise l'égalité sur les deux surfaces des formes $f_{q+1}, \cdots, f_{n-1}$ et des invariants $j_{p+1}, \cdots, j_{n-1}$ $(q \leq p)$. L'article se termine par une étude détaillée du cas $p=q=0$. *M. Decuyper* (Lille)

8451:

Júza, Miloslav. Sur les variétés représentant une généralisation des surfaces réglées. Czechoslovak Math. J. **10** (**85**) (1960), 440–456. (Russian summary)

L'auteur étudie les variétés à $n+1$ dimensions qui sont des systèmes à un paramètre d'espaces linéaires à n dimensions dans l'espace projectif de dimension $nk+k-1$. Étant donné dans cet espace $n+1$ courbes $y_0(v), \cdots, y_n(v)$, l'ensemble des points qu'on peut écrire sous la forme $x=t^iy_i(v)$ est appelé une variété non développable si

$$[y_0, \cdots, y_n, y_0', \cdots, y_n', \cdots, y_0^{(k-1)}, \cdots, y_n^{(k-1)}] \neq 0.$$

Les courbes $y_i(v)$ sont les courbes directrices, l'espace $[y_0(v), \cdots, y_n(v)]$ est l'espace générateur. Une courbe sur une telle variété non développable $V_{n,kn+k-1}$ est dite quasiasymptotique si en chacun de ses points son espace k-osculateur appartient à l'espace $(k-1)$-osculateur de la variété au même point; par chaque point de la variété passe une quasiasymptotique et une seule. L'auteur définit des systèmes normalisés de courbes directrices sur une variété non développable et prouve l'existence d'un paramètre v, appelé arc projectif, permettant une représentation plus simple de la variété. Dans ces conditions, on trouve un système d'homographies $H(v)$, $H_0(v), \cdots, H_{k-3}(v)$ et une fonction $j(v)$ indépendantes du choix du système normalisé de courbes directrices et qui, ensemble, déterminent uniquement, à des homographies près, la variété.

Pour terminer, l'auteur étudie les déformations projectives des variétés $V_{n,2n+1}$, en s'inspirant des travaux de E. Čech et A. Švec; il définit pour ces variétés une déformation projective spéciale du second ordre et obtient le théorème: Pour qu'une correspondance φ entre deux variétés non développables $V_{n,2n+1}$, $\bar{V}_{n,2n+1}$ qui met en homographie les espaces générateurs et conserve les courbes asymptotiques soit une déformation projective spéciale du second ordre, il faut et il suffit qu'elle conserve l'arc projectif et le système d'homographies déterminantes $H(v)$. *M. Decuyper* (Lille)

8452:

Lemleĭn, V. G. Induction of a connection of constant curvature in associated centro-projective spaces of a locally projective manifold. Dokl. Akad. Nauk SSSR **131** (1960), 17–20 (Russian); translated as Soviet Math. Dokl. **1**, 181–184.

L'auteur démontre d'abord un théorème établissant la condition pour que la connexion définie dans un espace centro-projectif $\{P_n\}$ par un objet $\gamma_{jk}{}^p(u)$ soit une connexion à courbure constante $K \neq 0$. D'autre part, étant donné un scalaire relatif de poids 1 dans un espace localement projectif, il associe à chaque espace centro-projectif local une connexion déterminée par un objet $\Gamma_{jk}{}^p(x, u)$ qui d'après le théorème précédent est à courbure constante. *M. Decuyper* (Lille)

8453:

Eisenhart, Luther P. Spaces which admit fields of normal null vectors. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1605–1608.

A study is made of n-dimensional Riemannian manifolds which admit fields of normal null-vectors. The latter are supposed to satisfy certain invariant differential equations in terms of their covariant derivatives. It appears that under these circumstances the Ricci tensor assumes certain forms, while the curvature scalar of the space vanishes identically. [See also #9243 and #9244 below.]

H. Rund (Durban)

8454:

Smith, J. Wolfgang. Lorentz structures on the plane. Trans. Amer. Math. Soc. **95** (1960), 226–237.

Consider a Lorentz plane V, that is, a C^∞ Lorentz fundamental tensor on the plane R^2. The author investigates the geometry of the time-like, space-like, and light-like curves in V. For example, if there is a diffeomorphism

of V onto R^2 which carries the two families of light-like geodesics of V to axis-parallel lines in R^2, then V is geodesically connected, that is, each pair of points of V can be joined by a geodesic of V. Also if V is complete and the curvature integral over V is absolutely convergent, then V is geodesically connected. The author then gives an example of a Lorentz plane V which is complete (for the linear connection) but which is not geodesically connected. *L. Markus* (Minneapolis, Minn.)

8455:

Abe, Shingo; Ikeda, Mineo. On groups of motions in space-time with a non-symmetric fundamental tensor $g_{\mu\nu}$. I. Tensor (N.S.) **10** (1960), 26–33.

The authors study groups of motions in a 4-dimensional space N_4 endowed with a non-symmetric fundamental tensor g_{ji}, the signature of the quadratic form $g_{ji}d\xi^j d\xi^i$ being -2. For an infinitesimal transformation $'\xi^h = \xi^h + v^h dt$, the generalized equations of Killing are given by

$$\underset{v}{\pounds}\, g_{ji} = v^a \nabla_a g_{ji} + g_{ai}\nabla_j v^a + g_{ja}\nabla_i v^a = 0,$$

where ∇_a denotes covariant derivative with respect to $h_{ji} = \frac{1}{2}(g_{ji}+g_{ij})$. The authors first prove: The generalized equations of Killing involves at least fourteen linearly independent components. Now put

$$g = |g_{ji}|/|h_{ji}|, \quad k = |k_{ji}|/|h_{ji}|,$$

where $k_{ji} = \frac{1}{2}(g_{ji} - g_{ij})$. The authors then prove: The number of linearly independent components of the generalized equations of Killing is as follows: (a) 14, when g and k are both constant; (b) 15, when only one of g and k is constant, or when the vectors $\nabla_i g$ and $\nabla_i k$ are parallel to each other; (c) 16, when neither of g and k is constant and the vectors $\nabla_i g$ and $\nabla_i k$ are not parallel to each other.

After having examined the integrability conditions of generalized equations of Killing, the authors prove: A group of motions in an N_4 has at most six parameters. In particular, in case (a) above, a group of motions in N_4 has at most six parameters; in case (b), at most five; in case (c), at most four. The maximum order of groups of motions in N_4 is equal to six. *K. Yano* (Seattle, Wash.)

8456:

Номидзу, К. [Nomizu, K.]. Группы Ли и дифференциальная геометрия. [Lie groups and differential geometry.] Translated from the English by Yu. A. Šub-Sizonenko; edited by G. F. Laptev. Izdat. Inostr. Lit., Moscow, 1960. 128 pp. 4.20 r.

For the original [Math. Soc. Japan, 1956] see MR **18**, 821. The present edition includes brief annotations by the translator and the editor.

8457:

Helgason, Sigurdur. Differential operators on homogenous spaces. Acta Math. **102** (1959), 239–299.

In this paper the author has been primarily interested in analytic properties based on the geometric structures of Riemannian and non-positive definite pseudo-Riemannian manifolds. For the sake of completeness the author has reviewed much known material. However, his presentation of the details of this review seems to this reader to be neat and in places quite original. In addition a great deal of detailed information has been added, which was previously not available. The following is the author's outline of the detailed content of his paper.

"Chapter I contains a general discussion of linear differential operators on manifolds. On pseudo-Riemannian manifolds there is always one differential operator, the Laplace-Beltrami operator, which is invariant under all isometries but under no other diffeomorphisms.

"Chapter II. In § 1 we recall some essentially known results on transitive transformation groups and homogeneous spaces. Two-point homogeneous spaces admit essentially only one invariant differential operator, the Laplace-Beltrami operator. Potential theory has a particularly explicit character. In § 3 we prove some properties of these spaces which are used later, e.g., the symmetry of non-compact two-point homogeneous spaces. A fairly direct proof of this fact is possible, but for the compact spaces such a proof seems to be unknown although the symmetry can be verified by means of Wang's classification. In § 4 we investigate in some detail Lorentzian spaces of constant curvature and the behavior of the geodesics on these spaces. For the spaces of negative curvature (simply connected) the time-like geodesics through a given point are infinite and do not intersect each other. The spaces of positive curvature that we consider have infinite cyclic fundamental group. Their time-like geodesics through a given point are all closed and do not intersect each other.

"Chapter III. In § 1 we represent the algebra $D(G/H)$ of invariant differential operators by means of the symmetric invariants of the group $\mathrm{Ad}_G(H)$. Thus if H is semi-simple, $D(G/H)$ has a finite system of generators. If G/K is a Riemannian symmetric space, $D(G/K)$ is finitely generated and commutative. For Lorentz spaces of constant curvature (or two-point homogeneous spaces) $D(G/H)$ is generated by the Laplace-Beltrami operator.

"Chapter IV. We consider in § 1 the mean value operators M^x which in a natural way generalize the operation M^r of averaging over spheres in R^n of fixed radius r. It is well known that M^r is formally a function (Bessel function) of the Laplacian Δ. The analogue holds for the space G/K if K is compact. In fact M^x is formally a function of the generators $D^1, \cdots, D^l$ of $D(G/K)$. This has some applications, for example, a generalization of the mean value theorem of Asgeirsson. For two-point homogeneous spaces we obtain more explicit results, for example, a simple geometric solution of Poisson's equation. In § 4 is given for a Riemannian space of constant curvature a decomposition of a function into integrals over totally geodesic submanifolds. A somewhat analogous problem is treated in § 7 for a Lorentzian space of constant curvature. Here a function is represented by means of its integrals over Lorentzian spheres. We use here methods of analytic continuation introduced by M. Riesz in his treatment of the wave equation. In § 8 we verify that Huygens' principle in Hadamard's formulation is absent for non-flat harmonic Lorentz spaces." *L. Auslander* (New Haven, Conn.)

8458:

Hermann, Robert. Existence in the large of totally geodesic submanifolds of Riemannian spaces. Bull. Amer. Math. Soc. **66** (1960), 59–61.

Let M be a complete Riemannian manifold, $m \in M$, and V a subspace of M_m. This paper sketches a proof of a theorem whose conclusion is that there exists a complete immersed totally geodesic submanifold through m whose tangent space there is V. Now we state the assumptions of this theorem. Call a once-broken geodesic admissible if and only if both: (a) its tangent vector at any point, when parallel-translated back to m, is in V; (b) the second geodesic segment lies in a convex neighborhood of the end point of the first geodesic segment. Let $R(x, y)$ be the Riemannian curvature transformation associated with an x and y in any tangent space to $M(R(x, y)$ is a transformation of this tangent space into itself). The statement of the theorem is: if σ is any admissible once-broken geodesic and V_σ the parallel translate of V along σ, then $R(x, y)z \in V_\sigma$ whenever $x, y, z \in V_\sigma$.

This extends a theorem of the reviewer [Ann. of Math. (2) **64** (1956), 337–363; MR **21** #1627].

W. Ambrose (Cambridge, Mass.)

8459:

O'Neill, Barrett. Immersion of manifolds of nonpositive curvature. Proc. Amer. Math. Soc. **11** (1960), 132–134.

Generalizing a theorem and proof of Tompkins [Duke Math. J. **5** (1939), 58–61] the following theorem is proved. Let M be a compact Riemannian manifold of dimension n, and M_1 a complete simply connected Riemannian manifold of dimension less than $2n$. If the Riemannian curvatures, K and K_1, of M and M_1 satisfy $K \leqq K_1 \leqq 0$, then M cannot be immersed in M_1.

W. Ambrose (Cambridge, Mass.)

8460:

Погорелов, А. В. [Pogorelov, A. V.]. ★Некоторые вопросы теории поверхностей в эллиптическом пространстве. [Topics in the theory of surfaces in an elliptic space]. Izdat. Har'kov. Gos. Univ., Kharkov, 1960. 92 pp. 4.25 r.

Let $x = (x_0, \cdots, x_3)$ be Weierstrass coordinates in the three-dimensional elliptic space E^3 and interpret these also as rectangular coordinates in four-dimensional euclidean space R^4. Let F', F'' be sets in E^3 lying in $x_0 \neq 0$, so that the coordinates of points in F', F'' can be uniquely determined by $x_0 > 0$. To a map $x' \to x''$ of F' on F'' we associate a map $y' \to y''$ of sets G', G'' in the euclidean subspace R^3 of R^4 by

$$y' = \frac{x' - e_0(x' \cdot e_0)}{e_0 \cdot (x' + x'')}, \quad y'' = \frac{x'' - e_0(x'' \cdot e_0)}{e_0 \cdot (x' + x'')}.$$

The booklet shows that by means of this map many deep theorems in R^3 can be extended to E^3 without new proofs. If $x' \to x''$ is an isometry, then so is $y' \to y''$; and conversely. A convex body in E^3 is defined as a set k for which a plane p disjoint from k exists and which is a convex body in the affine space obtained by considering p as the plane at infinity. A convex surface is an open connected subset of the boundary of a convex body. If F', F'' are intrinsically isometric closed convex surfaces, then so are G' and G''; since the latter are congruent, so are the former. The infinitesimal rigidity of closed convex surfaces without plane pieces in R^3 is shown to yield the same result for E^3, etc.

The author also proves the analogue to his theorem for R^3, namely, that a regular, not necessarily complete convex surface with E, F, G of class C^k, $k \geqq 5$, is at least of class C^{k-1}.

H. Busemann (Los Angeles, Calif.)

8461:

Šor, L. A. An example of a rigid surface which is a limit of surfaces isometric to it. Uspehi Mat. Nauk **15** (1960), no. 4 (94), 193–198. (Russian)

The paper provides an example of a convex surface S (in ordinary space) which is homeomorphic to a disc and admits no proper continuous isometric deformation ("cannot be bent"), but is the limit of a sequence of convex surfaces intrinsically isometric to S but not congruent to S.

H. Busemann (Los Angeles, Calif.)

8462:

Zaustinsky, Eugene M. Spaces with non-symmetric distance. Mem. Amer. Math. Soc. No. 34, 91 pp. (1959).

The theory of finitely compact metric spaces with locally unique geodesics has been systematically investigated by Busemann [*The geometry of geodesics*, Academic Press, New York, 1955; MR **17**, 779]. The aim of this booklet is to ascertain "just exactly to what extent" the results of Busemann's study depend upon the symmetry of the metric. It turns out "that nearly every theorem (so far attempted) of the symmetric theory which can be formulated at all without the assumption of symmetry holds without this assumption". The clearly expressed proofs are presented in great detail and are easily followed.

L. M. Blumenthal (Columbia, Mo.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 8048, 8102, 8163, 8365.

8463:

Kosiński, A. On some problems connected with the topology of manifolds. Proc. Internat. Congress Math. 1958, pp. 427–432. Cambridge Univ. Press, New York, 1960.

The author gives a brief survey of problems and known results concerning particularly the notion of "r-space". This is a generalisation of the notion of manifold, set-theoretic in character and distinct from the algebraic generalisations of Wilder and others.

H. B. Griffiths (Birmingham)

8464:

Mardešić, Sibe. On the Hahn-Mazurkiewicz theorem in nonmetric spaces. Proc. Amer. Math. Soc. **11** (1960), 929–937.

By an ordered continuum is meant a compact connected linearly ordered space with the order topology. A space X is connected by ordered continua if for each pair of points $x_0 \in X$ and $x_1 \in X$, there exist an ordered continuum C and a map $\chi: C \to X$ taking the end points of C into x_0 and x_1, respectively. A natural extension of the Hahn-Mazurkiewicz Theorem would be the result that every compact, connected, locally connected Hausdorff space is connected by ordered continua. The author constructs a counter-example to this conjecture.

E. Dyer (Chicago, Ill.)

8465:

Armentrout, Steve. Separation theorems for some plane-like spaces. Trans. Amer. Math. Soc. **97** (1960), 120–130.

The paper deals with separation theorems like the following one of Z. Janiszewski. Let S be the Euclidean plane, and let $H\subset S$, $K\subset S$ be (compact) continua neither of which separates two points $x, y\in S$. If $H\cap K=\emptyset$ or $H\cap K$ is connected, then $H\cup K$ does not separate x and y. R. L. Moore proved these theorems for "plane-like" spaces S satisfying his axioms 0–5 [*Foundations of point set theory*, Amer. Math. Soc., New York, 1932; Theorems 16 (p. 187) and 18 (p. 190)]. The author establishes the same theorems for a different class of "plane-like" spaces S. S is assumed to be a connected and locally compactly connected metric space with no cut points (S need not be complete). Two more conditions are imposed on S; these are analogues of Moore axioms 4 and 5, where chains serve as substitutes for arcs. A chain is defined as a finite sequence of continua g_i, where only the consecutive ones intersect and $g_i\cap g_{i+1}$ contains exactly one point. An example shows that such a space need not contain arcs or simple closed curves. *S. Mardešić* (Zagreb)

8466:

Jones, F. Burton. Another cutpoint theorem for plane continua. Proc. Amer. Math. Soc. **11** (1960), 556–558.

If p and q are points of a continuum M and x is a point of $M-\{p,q\}$, x is said to cut p from q in M if every subcontinuum of M which contains p and q contains x. Thus, while no point of an indecomposable continuum separates it, every point of it cuts it. The author improves his previous result [Proc. Amer. Math. Soc. **9** (1958), 530–532; MR **20** #3515] by removing the restriction on pairs of points. His theorem now is: Suppose that M is a compact subcontinuum of the plane S which does not separate S. If no point of M cuts the point p from the point q in M, then some component of the set of interior points of M contains both p and q in its closure. A counterexample is given to the converse.

J. Segal (Seattle, Wash.)

8467:

Ward, L. E., Jr. On local trees. Proc. Amer. Math. Soc. **11** (1960), 940–944.

Characterizations of continua which are locally acyclic (or dendritic) are given in three forms: (1) as locally connected continua containing no small simple closed curves; (2) as 1-dimensional absolute neighborhood retracts; and (3) as continua admitting a partial ordering in terms of their cut points of a type previously studied by the author. *G. T. Whyburn* (Charlottesville, Va.)

8468:

Bing, R. H.; Curtis, M. L. Imbedding decompositions of E^3 in E^4. Proc. Amer. Math. Soc. **11** (1960), 149–155.

If a monotone decomposition of E^3 has only finitely many nondegenerate elements C_i, and if when one considers E^3 as a 3-plane in E^4 there exist points p_i in E^4-E^3 such that the cones over C_i from p_i are disjoint, then the decomposition space is imbeddable in E^4. In particular, if the nondegenerate elements either (1) are two in number or (2) consist of the three linked, pairwise unlinked, circles ("Ballantine Ale" design), then the decomposition space is imbeddable in E^4. However, an example is given of a decomposition of E^3 whose nondegenerate elements are nine circles and whose decomposition space is not imbeddable in E^4. The well-known Flores complex P is utilized and a new proof, based on work of Shapiro, is given for the non-imbeddability of P in E^4.

R. L. Wilder (Ann Arbor, Mich.)

8469:

Mazur, Barry. On embeddings of spheres. Bull. Amer. Math. Soc. **65** (1959), 59–65.

The Schoenflies problem classically asks whether the closures of the complements of an imbedded $(n-1)$-sphere S^{n-1} in an n-sphere S^n are homeomorphic to n-cells. One also has the combinatorial and differentiable versions, with the imbedding and homeomorphism combinatorial in one case and differentiable in the other. The Alexander horned sphere of an imbedding of S^2 in E^3 shows the answer in the topological case to be false. Alexander (and subsequent improvements on his methods) gave an affirmative answer in the combinatorial and differentiable case for S^2 in E^3. The author's work represents the next important step in this problem by giving a partial answer for arbitrary n. This is as follows. Suppose $\psi: S^{n-1}\to S^n$ is an imbedding such that (a) ψ can be extended to a homeomorphism ϕ of $S^{n-1}\times[-1,1]$ and (b) ϕ is semi-linear in the neighborhood of some point in $S^{n-1}\times[-1,1]$ with respect to a subdivision of the usual simplicial structure on S^n. If, for example, ψ is a differentiable imbedding then (a) and (b) are satisfied. The main theorem says that if an imbedding $\psi: S^{n-1}\to S^n$ satisfies (a) and (b), then the closures of the complements of $\psi(S^{n-1})$ in S^n are homeomorphic to the n-cell. The proof is quite novel and elementary. Since this paper by the author there has been much further progress on the Schoenflies problem.

S. Smale (Berkeley, Calif.)

8470a:

Morse, Marston. A reduction of the Schoenflies extension problem. Bull. Amer. Math. Soc. **66** (1960), 113–115.

8470b:

Brown, Morton. A proof of the generalized Schoenflies theorem. Bull. Amer. Math. Soc. **66** (1960), 74–76.

Suppose a homeomorphic embedding h of $S^{n-1}\times[0,1]$ in S^n is given. Are the closures of the complementary domains of $h(S^{n-1}\times\frac{1}{2})$ topological n-cells? Mazur [#8469 above] gave an affirmative answer if h is "nice" in the neighborhood of at least one point of $S^n\times\frac{1}{2}$. Both the present papers show that this "niceness" condition may be dropped. Morse shows that the theorem with the "niceness" condition implies already the theorem without that condition. Brown achieves it by giving a completely new and very elegant and brief proof of the theorem.

S. Eilenberg (New York)

8471:

Černavskiĭ. A. V. Twofold continuous partitions of a ball. Dokl. Akad. Nauk SSSR **131** (1960), 1272–1275 (Russian); translated as Soviet Math. Dokl. **1**, 436–439.

It is proved that in every continuous decomposition of the ball Q^n ($n\le 3$) into pairs of points, there is at least one pair whose elements coincide. (This was proved for

$n=1$ by O. G. Harrold [Duke Math. J. 5 (1939), 789–793; MR 1, 223] and for $n=2$ by J. H. Roberts [ibid. 6 (1940), 256–262; MR 1, 319].) *P. A. Smith* (New York)

8472:

Fort, M. K., Jr. Neighborhood extensions of continuous selections. Proc. Amer. Math. Soc. **11** (1960), 682–685.

Let F be a function from a paracompact space X to the space of subsets of a topological space Y. Call $T\subset Y$ an F-neighborhood extension set (F-NES) with respect to a closed $S\subset X$ if, for all closed $A\subset S$, every continuous $f: A\to T$, with $f(x)\in F(x)$ for all $x\in A$, can be extended to a continuous $g: U\to T$, with $U\supset A$ open and $g(x)\in F(x)$ for all $x\in A$. Theorem (the author's theorem 1 is slightly sharper): If, for every $x\in X$ and $y\in Y$, there exist neighborhoods V_x in X and U_y in Y such that U_y is an F-NES with respect to $\bar{V}_x$, then Y is an F-NES with respect to X. If $F(x)=Y$ and $V_x=X$ for all $x\in X$, this was first proved by Hanner [Ark. Mat. **2** (1952), 315–360; MR **14**, 396; Theorem 19.2], whose proof was longer than the author's. Theorem 2 gives an application to fibre bundles. *E. Michael* (Princeton, N.J.)

8473:

Ponomarev, V. I. A new space of closed sets and many-valued continuous mappings of bicompacts. Mat. Sb. (N.S.) **48 (90)** (1959), 191–212. (Russian)

Details of results previously announced in Dokl. Akad. Nauk SSSR **118** (1958), 1081–1084 [MR **20** #6685]. The author also compares his notion of continuity for many-valued mappings with that of Strother [Duke Math. J. **22** (1955), 551–556; MR **17**, 288], which he calls "strong continuity". *J. R. Isbell* (Seattle, Wash.)

8474:

Halfar, Edwin. Conditions implying continuity of functions. Proc. Amer. Math. Soc. **11** (1960), 688–691.

The author presents six theorems, of which the following two are typical. (1) Let f be a function from a locally compact Hausdorff space X onto a Hausdorff space Y. If $f(K)$ is compact for every compact set $K\subset X$, and if $f^{-1}(y)$ is closed for every point $y\in Y$, then f is continuous on X. (2) Let X be a locally connected Hausdorff space with the following property: whenever an infinite set $A\subset X$ has x as an accumulation point, there is a compact subset of $A\cup\{x\}$ which has x as an accumulation point. Let f be a function from X onto a Hausdorff space Y. If $f(K)$ is compact for every compact set $K\subset X$, and if $f(C)$ is connected for every connected set $C\subset X$, then f is continuous on X. *Ky Fan* (Detroit, Mich.)

8475:

Hamstrom, Mary-Elizabeth. Regular mappings whose inverses are 3-cells. Amer. J. Math. **82** (1960), 393–429.

This paper contains extensions, of the type indicated in the title, of earlier work of the author and Eldon Dyer concerned with regularity (in various dimensions) of convergence of sequences and of mappings, and develops conditions under which such mappings approximate fibre and projection maps. The natural formulation of regularity conditions in terms of homotopy rather than homology plays a central rôle in this work. Among other results it is shown that if f is a compact homotopy 2-regular mapping of one metric space, X, onto another, Y, such that point inverses under f are all 3-cells, then f is completely regular in the sense that for any $\varepsilon>0$ and $y\in Y$ a $\delta>0$ exists such that if y' is any point of y at distance $<\delta$ from y then $f^{-1}(y)$ and $f^{-1}(y')$ are ε-homeomorphic. A similar conclusion is obtained in case the point inverses are compact 3-manifolds imbeddable in E_3 and having boundaries which are mutually homeomorphic. *G. T. Whyburn* (Charlottesville, Va.)

8476:

Whyburn, G. T. Convergence in norm. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1614–1617.

Let X, Y, be metric, and $f_n: X\to Y$ a sequence of onto maps converging to an onto map $f: X\to Y$. The author is interested in determining properties of f under the additional condition that the norms

$$N(f_n) = \sup\{\operatorname{diam} f_n^{-1}(y) | y\in Y\}\to 0.$$

Some stated consequences of his general results: (a) X compact, Y locally connected, convergence uniform, and $N(f_n)\to 0$ imply f is monotone. (b) X, Y locally connected generalized continua, X cyclic, convergence pointwise, and $N(f_n)\to 0$ imply Y is cyclic. (c) If $X=Y=S^2$, then all monotone onto maps, and only monotone onto maps, can be uniform limits of sequences of onto maps with $N(f_n)\to 0$. (d) X, Y compact, convergence uniform, and the sequence $\{f_n\}$ uniformly approximately open implies f is a homeomorphism. {There are numerous misprints on p. 1616.} *J. Dugundji* (Los Angeles, Calif.)

8477:

Bagley, R. W. On pseudo-compact spaces and convergence of sequences of continuous functions. Proc. Japan Acad. **36** (1960), 102–105.

Supplementing previous characterizations, the author shows that an arbitrary space is pseudocompact if and only if every sequence of continuous real functions which is convergent in some jointly continuous topology for the function space is uniformly convergent. Under quite weak countability assumptions, "some jointly continuous" can be replaced by "the compact-open". The result of Exercise 51 of L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960 [MR **22** #6994] is posed as an unsolved problem. *J. R. Isbell* (Seattle, Wash.)

8478:

Iséki, Kiyoshi. Countable compactness and quasi-uniform convergence. Proc. Japan Acad. **36** (1960), 187–188.

Countably compact spaces are characterized by a condition involving convergence of sequences of upper semi-continuous functions. *J. R. Isbell* (Seattle, Wash.)

8479:

Mrówka, S. Compactness and product spaces. Colloq. Math. **7** (1959), 19–22.

A proof is given of the following theorem [cf. J. Novák, Fund. Math. **40** (1953), 106–112; MR **15**, 640; C. Ryll-Nardzewski, Bull. Acad. Polon. Sci. Cl. III **2** (1954),

265–266; MR 16, 157]: If X is countably compact, Y is either compact or sequentially compact, then $X \times Y$ is countably compact. The author poses the problem of finding conditions on X such that Y countably compact implies $X \times Y$ countably compact. {This problem has been examined recently by Z. Frolík [#8480 below].} It is proved that if X is pseudocompact [resp., Lindelöf], Y compact, then $X \times Y$ is pseudocompact [resp., Lindelöf]; this result is known [cf., e.g., E. Čech, *Topologické prostory*, Naklad. Československé Akad. Věd, Prague, 1959; MR 21 #2962; p. 445; and N. Bourbaki, *Topologie générale*, Chap. 9, 2nd ed., Actualités Sci. Ind. no. 1045, Hermann, Paris, 1958, p. 103; 1st ed., MR 10, 260].

Katětov (Prague)

8480:

Frolík, Zdeněk. The topological product of countably compact spaces. Czechoslovak Math. J. **10** (85) (1960), 329–338. (Russian summary)

The author generalizes from (1) Novák's example [Fund. Math. **40** (1953), 106–112; MR 15, 640] of a product of two countably compact spaces containing N as a closed subspace, and (2) the theorem that a product of two countably compact spaces is countably compact if one of them is first countable. In (1), N can be replaced by any discrete space or any separable metrizable space. In (2), first countability is weakened to a necessary and sufficient condition (on countably compact P, in order that $P \times Q$ is countably compact whenever Q is); it is sufficient that every infinite discrete set has an infinite subset with only finitely many accumulation points, or that P can be compactified by the addition of countably many points.

J. R. Isbell (Seattle, Wash.)

8481:

Parnaby, T. W. Some order properties of coverings of finite-dimensional spaces. Proc. Glasgow Math. Assoc. **4**, 188–197 (1960).

For a covering $\mathfrak{U}$ of a space X, the order function $x:\mathfrak{U}$ is defined as the number of members of $\mathfrak{U}$ which contain x. The main theorem is that each normal X with $\dim X \geq n$ has a finite open covering such that for any locally finite open or closed refinement $\mathfrak{U}$, $x:\mathfrak{U}$ assumes at least $n+1$ distinct values. Indeed, on some member of $\mathfrak{U}$ the order function assumes at least $n+1$ distinct values; and if X is paracompact, this happens on every neighborhood of some point. A family of special cases is taken up, sharpened, and each is shown to characterize dimension.

J. R. Isbell (Seattle, Wash.)

8482:

Boltyanskiĭ, V. G. Mappings of compacta into Euclidean spaces. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 871–892. (Russian)

A mapping f of a compactum X into a euclidean space E was called by Borsuk k-regular if for any distinct points $x_0, x_1, \cdots, x_k$ of X the points $f(x_0), f(x_1), \cdots, f(x_k)$ do not lie in any $(k-1)$-dimensional subspace. The author proves that the k-regular mappings of X into E are dense in the space E^X whenever $1+\dim E \geq (1+k)(1+\dim X)$, whereas if $1+\dim E < (1+k)(1+\dim X)$ this need not be so.

R. H. Fox (Princeton, N.J.)

8483:

Dauker, K. H. [Dowker, C. H.]. Affine and euclidean complexes. Dokl. Akad. Nauk SSSR **128** (1959), 655–656. (Russian)

Continuing his previous work [Amer. J. Math. **74** (1952), 555–577; MR 13, 965] the author considers Euclidean complexes, i.e., metric complexes whose cells have the usual Euclidean metric. He announces that a Euclidean complex has a subdivision which is a simplicial Euclidean complex, and that isomorphic Euclidean complexes have subdivisions which are isomorphic simplicial Euclidean complexes. One corollary is that isomorphic Euclidean complexes are homeomorphic.

D. W. Kahn (New Haven, Conn.)

8484:

Smart, D. R. A fixed-point theorem. Proc. Cambridge Philos. Soc. **57** (1961), 430.

Proof of the theorem: In a compact metric space in which the identity mapping can be uniformly approximated by contraction mappings, every subisometric mapping [=mapping T such that $\rho(Tx, Ty) \leq \rho(x, y)$] has a fixed point.

8485:

Stallings, J. Fixed point theorems for connectivity maps. Fund. Math. **47** (1959), 249–263.

Let f be a transformation $X \to Y$ with graph $\Gamma(f)$ in $X \times Y$. Then f is a connectivity map if $\Gamma(f|C)$ is connected when C is connected. It is almost continuous (a.c.) if $\Gamma(f)$ in an open set N implies that $\Gamma(g) \subset N$ for some continuous map $g: X \to Y$. Local definitions arise by going to neighborhoods, polyhedral properties by restricting the spaces to be concrete simplicial complexes and taking star neighborhoods. Relations are established between these concepts and a literature proof is corrected. Fixed points exist for a.c. maps if they exist for continuous maps.

D. G. Bourgin (Urbana, Ill.)

8486:

Young, G. S. Fixed-point theorems for arcwise connected continua. Proc. Amer. Math. Soc. **11** (1960), 880–884.

As a consequence of an earlier theorem of the author [Amer. J. Math. **68** (1946), 479–494; MR 8, 49] a rather technical necessary condition is given for an arcwise connected, compact Hausdorff space not to have the fixed point property. From this are easily derived some previously known results in fixed point theory together with the theorem that every contractible Hausdorff continuum each two points of which lies on a unique arc has the fixed point property. An example is given of an arcwise connected continuum which contains no simple closed curve but does not have the fixed point property.

E. Dyer (Chicago, Ill.)

8487:

Fisher, Gordon M. On the group of all homeomorphisms of a manifold. Trans. Amer. Math. Soc. **97** (1960), 193–212.

The author studies the topological group $G(M)$ of all homeomorphisms of a closed manifold M onto itself. If the dimension of M is less than 4, he shows the identity component of $G(M)$ is algebraically simple, equals the

group of deformations of M (the group of all homeomorphisms of M onto itself which are isotopic to the identity), and is open in $G(M)$. The proofs in dimensions 2 and 3 are given separately; the latter case uses heavily techniques and results of Bing and Moise.

E. Dyer (Chicago, Ill.)

8488:

Jakobsen, J. F.; Utz, W. R. The non-existence of expansive homeomorphisms on a closed 2-cell. Pacific J. Math. **10** (1960), 1319–1321.

A homeomorphism T of a metric space X onto itself is said to be expansive provided there exists a positive real number a such that $x, y \in X$ with $x \neq y$ implies the existence of an integer n for which $\operatorname{dist}(T^n x, T^n y) > a$. For example, the shift transformation in symbolic dynamics is expansive and acts upon a space homeomorphic to the Cantor discontinuum; actually, this is the historical origin of the property. The authors prove that no homeomorphism of a closed 2-cell onto itself is expansive by showing that no homeomorphism of a simple closed curve onto itself is expansive. They also announce without proof that the shift transformation on the inverse limit space of any continuous map of an arc onto itself cannot be expansive.

It has been previously observed by several workers independently that an arc cannot carry an expansive homeomorphism. The question of whether a closed n-cell can carry an expansive homeomorphism has thus a negative answer for $n=1$ and $n=2$; the question remains open for $n>2$.

W. H. Gottschalk (New Haven, Conn.)

8489:

Beck, Anatole. Continuous flows with closed orbits. Bull. Amer. Math. Soc. **66** (1960), 305–307.

The author announces a topological characterization of those subsets F of the plane such that F is the fixed-point set of some continuous flow in the plane all of whose orbits are closed subsets of the plane. A similar characterization with "closed" replaced by "compact" is also given.

W. H. Gottschalk (New Haven, Conn.)

8490:

Hahn, F. J. Recursion of set trajectories in a transformation group. Proc. Amer. Math. Soc. **11** (1960), 527–532.

Let (X, T) be a transformation group, where X is a metric space and T is a topological group. Allowing T to act on the subsets of X, the author discusses various questions of continuity and recursion thus suggested. Equivariant mappings, which to a point x of X assign a closed subset of X containing x and which commute with the transitions of (X, T), are also studied. In particular, several theorems of the form "regional recursion implies the recursive points form a residual set" are proved. The author's results establish a conjecture of Trjitzinsky. Examples of flows are presented to show that certain variations in hypothesis and conclusion are not possible.

W. H. Gottschalk (New Haven, Conn.)

8491:

Ellis, Robert. Universal minimal sets. Proc. Amer. Math. Soc. **11** (1960), 540–543.

Let T be a discrete group. A universal minimal set associated with T is defined to be a minimal transformation group (M, T) with compact Hausdorff phase space M such that every minimal transformation group (X, T) with compact Hausdorff phase space X is a homomorphic image of (M, T). The author proves that (M, T) exists, has unique isomorphism type, and is strongly effective. As an application, he shows that the points of T are separated by almost periodic functions of a weakened kind assuming only two values.

W. H. Gottschalk (New Haven, Conn.)

8492:

Griffin, John S., Jr. On the rotation number of a normal curve. Compositio Math. **13**, 270–276 (1958).

The author studies the relation between the rotation number of a closed curve in the plane and the topological index of the curve. For normal curves f, i.e., curves with only a finite number of multiple points, it is shown that a decomposition into simple circuits exists such that 2π times the sum of the orientation numbers of these circuits is the rotation number of f, and this is expressible simply in terms of the topological indices of these circuits about any point not on f.

G. T. Whyburn (Charlottesville, Va.)

8493:

Plunkett, Robert L. Openness of the derivative of a complex function. Proc. Amer. Math. Soc. **11** (1960), 671–675.

The strong openness of the derivative mapping for a non-constant differentiable complex-valued function in a region of the complex plane is established by topological methods independent of the existence of a second derivative. The continuity of the derivative has been proven earlier by the author using similar methods. Uniform differentiability of such a function on the compact subsets of the region is first proven in this same setting. Then two proofs for strong openness of the derivative are given. The first obtains the conclusion by translating the uniform differentiability just established into uniform convergence of a sequence of differential quotients and then applying known theorems of topological analysis. The second procedure develops sharper results concerning the topological index of the derivative mapping and applies these to show that the latter also maps a closed Jordan subregion onto a set consisting of the image of the boundary together with bounded components of the complement of this image.

G. T. Whyburn (Charlottesville, Va.)

8494:

Reichbach, M. Some theorems on mappings onto. Pacific J. Math. **10** (1960), 1397–1407.

Earlier results of the author related to the onto property of a mapping are extended to general topological spaces. Roughly speaking, a mapping f of X into Y is called a polynomial mapping if it maps every sequence in X not containing a convergent subsequence onto a similar sort of sequence in Y. It is shown that if X is complete, any such mapping is closed, so that in case the image set $f(X)$ is open in Y and Y is connected, $f(X)$ must be all of Y. The same conclusion is obtained when $f(X)$ is assumed open only at points of $f(X)-J$, where J is a (proper) sub-

set of Y which does not disconnect Y. The concept of a lower bounded rate of change is studied as a substitute for the openness requirement on the mapping and, on this basis, still further generalizations are made of the Fundamental Theorem of Algebra. Results apply, in particular in an n-dimensional Banach space, $n \geqq 2$, when J can be taken as a countable set. For differentiable mappings, the set J in general corresponds to the image of the set of zeros of the Jacobian.

G. T. Whyburn (Charlottesville, Va.)

8495:

Fichera, Gaetano. Teoria astratta del prolungamento di Weierstrass e applicazioni. Ann. Mat. Pura Appl. (4) **49** (1960), 213–227.

Il s'agit d'une intéressante synthèse abstraite de plusieurs théories de prolongement fonctionnel, dont le prolongement analytique de Weierstrass est le prototype. Soient Σ un espace métrique et S un ensemble abstrait; en considérant les fonctions f définies dans des boules B de Σ et à valeurs dans S, l'auteur appelle "éléments fonctionnels" $\Sigma \to S$ les couples (B, f). Une "classe de Weierstrass" sera toute classe $\mathfrak{C}$ d'éléments fonctionnels vérifiant les conditions: (α) si deux éléments (B, f) et (B^*, f^*) sont tels que $B \cap B^*$ n'est pas vide et $f(x) = f^*(x)$ sur un ensemble ouvert contenu dans $B \cap B^*$, alors $f(x) = f^*(x)$ sur $B \cap B^*$; (β) si $(B, f) \in \mathfrak{C}$ et $B^* \subset B$, alors $(B^*, f) \in \mathfrak{C}$. Les classes de Weierstrass étudiées par l'auteur vérifient encore une condition supplémentaire. Cela posé, on définit "prolongement direct" d'un élément fonctionnel, comme dans le cas des fonctions analytiques; on en déduit une relation d'équivalence, qui détermine une partition de $\mathfrak{C}$ en classes d'équivalence, nommées "fonctions analyticoïdes $\Sigma \to S$". Les notions de "domaine naturel d'existence", "fonctions uniformes ou pluriformes", etc., de la théorie classique de Weierstrass, s'étendent aussitôt au cas des fonctions analyticoïdes. L'auteur fait une étude soignée du prolongement suivant une ligne, ce qui lui permet de généraliser aux fonctions analyticoïdes la notion de point singulier et le théorème de monodromie.

Comme première application de cette théorie générale, l'auteur démontre le théorème suivant de topologie globale: Toute application continue et localement inversible, T, d'un espace topologique S dans un espace métrique Σ est biunivoque, si les conditions suivantes sont vérifiées: (1) S est séparé, connexe et localement compact dénombrable à l'infini; (2) Σ est "à connexion linéaire simple"; (3) pour tout couple de boules B et B^* de Σ, il existe une ligne contenue $B \cup B^*$ dont les extrêmes sont les centres de B et B^*.

Ensuite, l'auteur applique sa théorie au cas des fonctions analytiques sur une variété différentiable de classe C^∞ et retrouve, comme cas très particulier, la théorie de Weierstrass relative aux fonctions analytiques complexes sur une surface de Riemann.

Une troisième application concerne l'intégrabilité globale d'un système aux différentielles totales sur une variété différentiable réelle V: si le système vérifie la condition naturelle (nécessaire) d'intégrabilité locale et encore une certaine majoration, par rapport à chaque carte de V, et si, d'autre part, V est à connexion linéaire simple, alors il existe une et seulement une solution du système, prenant des valeurs données en un point quelconque de V.

J. Sebastião e Silva (Lisbon)

ALGEBRAIC TOPOLOGY

See also 8483.

8496:

Fan, Ky. Combinatorial properties of certain simplicial and cubical vertex maps. Arch. Math. **11** (1960), 368–377.

A simplicial n-dimensional pseudomanifold here means a complex K^n consisting of a set of n-simplexes, and all their faces, each $(n-1)$-simplex being incident with at most two n-simplexes (no connectedness required). Let ϕ map the vertices of K^n into the $2m$ integers $(\pm 1, \cdots, \pm m)$, for a given m, so that $\phi(u) + \phi(v) \neq 0$ if u, v are adjacent, that is, joined by a 1-simplex. For each combination $(j_0, \cdots, j_n)$ of $n+1$ not necessarily distinct numbers from $(\pm 1, \cdots, \pm m)$, let $\alpha(j_0, \cdots, j_n)$ be the number of n-simplexes whose vertices, in some order, map onto $(j_0, \cdots, j_n)$; and let $\beta(j_0, \cdots, j_{n-1})$ be similarly defined for the $(n-1)$-simplexes. Theorem 1:

$$\begin{aligned}\textstyle\sum' \{\alpha(k_0, -k_1, k_2, -k_3, \cdots, (-1)^n k_n) \\ + \alpha(-k_0, k_1, -k_2, k_3, \cdots, (-1)^{n-1} k_n)\} = \\ \textstyle\sum'' \beta(k_0, -k_1, \cdots, (-1)^{n-1} k_{n-1}), \bmod 2,\end{aligned}$$

where $\sum'$ and $\sum''$ are sums with $1 \leqq k_0 \leqq k_1 \leqq \cdots$. This theorem, applicable to all pseudomanifolds, unifies Sperner's lemma and a combinatorial property of n-spheres previously established by the author [Ann. of Math. (2) **56** (1952), 431–435; MR **14**, 490]. Theorems 2 and 3, deduced from theorem 1, generalize the combinatorial property and Sperner's lemma, respectively. Analogues of theorems 1–3 are established for cubical pseudomanifolds, in which cubes replace simplexes as elements.

S. S. Cairns (Urbana, Ill.)

8497:

Bokštein, M. F. Complete modular spectrum of cohomology rings of a Tyhonov product. Mat. Sb. (N.S.) **51** (**93**) (1960), 73–98; erratum, **53** (**95**) (1961), 261–263. (Russian)

In previous papers the author has introduced the notion of the complete modular spectrum of cohomology rings of a locally compact space—roughly, the cohomology rings with coefficients in Z and Z_m for each integer m, together with homomorphisms between these rings induced by obvious coefficient homomorphisms and finally certain Bockstein (connecting) homomorphisms arising from short exact sequences. He has shown that this spectrum for a locally compact space X determines the cohomology ring of X with any field of coefficients, and also that the spectrum of a product of a finite number of locally compact spaces is determined by the spectra of the factors. In the present paper he shows that the spectrum of an arbitrary product of compact spaces is determined by the spectra of the factors.

N. Stein (New York)

8498:

Swan, Richard G. The homology of cyclic products. Bull. Amer. Math. Soc. **65** (1959), 125–127.

Announcement of results of #8499.

8499:

Swan, Richard G. The homology of cyclic products. Trans. Amer. Math. Soc. **95** (1960), 27–68.

L'auteur commence par développer la théorie des applications universelles dans le cadre des catégories et foncteurs, en particulier dans les catégories munies d'une notion d'homotopie (on a alors des applications qui sont universelles à une homotopie près). Soient alors X un complexe semi-simplicial, et Π le groupe cyclique d'ordre n opérant sur X^n par permutations circulaires des facteurs; le complexe $CP^nX = X^n/\Pi$ est appelé le produit cyclique n-uple de X. Dold a montré que l'homologie de CP^nX est déterminée par celle de X [Ann. of Math. (2) **68** (1958), 54–80; MR **20** #3537]. On donne ici une méthode permettant de calculer explicitement cette homologie. Cette construction est fonctorielle à une homotopie près, et permet (si X est muni d'une application diagonale) de calculer les cup-produits relatifs à CP^nX.

P. Samuel (Clermont-Ferrand)

8500:

Adams, J. Frank. Théorie de l'homotopie stable. Bull. Soc. Math. France **87** (1959), 277–280.

Let S^nX denote the n-fold (reduced) suspension of a finite CW complex X and $\pi_m{}^s(X, Y) = \lim_{n\to\infty} \pi(S^{m+n}X, S^nY)$, where π denotes the track-group of homotopy-classes; the $\pi_m{}^s$ are the "stable" track-groups. The author describes a spectral sequence

$$\mathrm{Ext}_A\,(H^*(Y, Z_p); H^*(X, Z_p)) \Rightarrow {}_p\pi_*{}^s(X, Y),$$

where A is the Steenrod algebra and ${}_p\pi$ the quotient group of π modulo elements of order prime to p. The sequence is obtained by forming a "geometric realisation" of a resolution

$$H^*(Y) \xleftarrow{\varepsilon} C_0 \xleftarrow{d_1} C_1 \leftarrow \cdots$$

over the Steenrod algebra. The successive differentials of the spectral sequence are then related to the successive orders of cohomological operations; only very brief indications appear in this paper.

V. Gugenheim (Baltimore, Md.)

8501:

Shih, Weishu. Sur les complexes généralisés et la suite spectrale d'un fibré. Bull. Soc. Math. France **87** (1959), 451–453.

Let E be an object in an abelian category and $f_i \in \mathrm{Hom}(E, E)$ $(i \geq 0)$ a sequence of morphisms such that

$$f_{i+1}Z_i \subset Z_i, \quad f_{i+1}B_i \subset B_i, \quad f_{i+1}{}^2Z_i \subset B_i \quad (i \geq 1),$$

where

$$Z_{-1} = E, \qquad B_{-1} = 0,$$
$$Z_{i+1} = f_{i+1}^{-1}B_i \cap Z_i, \quad B_{i+1} = (f_{i+1}Z_i) \cap Z_i + B_i \qquad (i \geq -1).$$

The assembly (E, f_i) is called a generalised complex. We define $H_i(E) = Z_i/B_i$ so that f_{i+1} induces $\bar f_{i+1} \in \mathrm{Hom}(H_i(E), H_i(E))$, a differential whose homology group is isomorphic to $H_{i+1}(E)$. The sequence $\{H_i(E)\}$ thus obtained is called the spectral sequence of the generalised complex. Theorem: Let F, B be the fibre and base space of a fibre space; then we can give to $E = C(F) \otimes C(B)$ the structure of a generalised complex such that its spectral sequence is isomorphic to the classical spectral sequence.

The author also indicates that the maps f_i are given by a suitable cap-product; these results (obtained independently) are of course closely related to the theorem on "twisted tensor products" due to E. H. Brown.

V. Gugenheim (Baltimore, Md.)

8502:

Shih, Weishu. Une propriété de la classe caractéristique d'un fibré principal. C. R. Acad. Sci. Paris **251** (1960), 1331–1332.

Let G be an $(n-1)$-connected topological group, $p: E \to B$ a principal G-bundle and $\xi \in H^{n+1}(B; H_n(G))$ the characteristic class of E. The main result of this note is that $0 = \xi^2 \in H^{2n+2}(B; H_{2n}(G))$, where the cup product is formed with respect to the pairing of $H_n(G)$, $H_n(G)$ to $H_{2n}(G)$ obtained from Pontryagin multiplication. As indicated in this note, a proof can be easily obtained by considering the bar construction applied to the singular chains of G.

E. H. Brown (Waltham, Mass.)

8503:

Koszul, Jean-Louis. Complexes d'espaces topologiques. Bull. Soc. Math. France **87** (1959), 403–408.

L'auteur reprend l'essentiel des résultats exposés dans un travail antérieur [Nagoya Math. J. **15** (1959), 155–169; MR **22** #982], et les précise dans le cas suivant: le fibré principal étudié est l'espace C^n, le groupe Γ est un sous-groupe discret de rang maximum de C^n, et le groupe G est un groupe abélien connexe, quotient de C^m par un sous-groupe discret D. Il considère notamment sur Y les "facteurs" $Y \times \Gamma \to G$ du type "thêta", c'est-à-dire, de la forme

$$(y, s) \to \exp(L(y, s) + H(s)),$$

où L est une application R-bilinéaire $Y^2 \to C^m$ telle que $L(iy, y') = iL(y, y')$. Pour qu'un fibré ayant un tel facteur soit associé [resp. préassocié] à Y, il faut et il suffit que L soit symétrique [resp. C-bilinéaire].

(Erratum: p. 407, ligne 5, au lieu de C et D^m, lire C^m et D.)

H. Cartan (Paris)

8504:

Crawley, Peter. On the equivalence of two surfaces. Amer. Math. Monthly **67** (1960), 165–166.

Informal argument with diagrams illustrating equivalence between a Klein bottle with cross-cap and a torus with cross-cap.

8505:

Gluck, Herman. The weak Hauptvermutung for cells and spheres. Bull. Amer. Math. Soc. **66** (1960), 282–284.

The following theorem is proved: If P and Q are two triangulations of the n-sphere (closed n-cell), there is a third triangulation M which can be obtained from either by subdivision. In fact, M can be obtained from either P or Q by subdivision of a single n-simplex. The author notes that this does not prove P combinatorially equivalent to Q.

S. S. Cairns (Urbana, Ill.)

8506:

Poénaru, Valentin. Sur les variétés simplement connexes à trois dimensions. Rend. Mat. e Appl. (5) **18** (1959), 25–94.

Let K be a triangulated closed simply connected 3-dimensional manifold and let I_j and E_j denote the j-dimensional closed cell and the j-dimensional euclidean space respectively. The author proves that (A) if P is a point of K, then $(K-P)\times E_2$ is homeomorphic to E_5, and (C) if I_3 is semilinearly imbedded in K, then $(K-\operatorname{int} I_3)\times I_2$ is homeomorphic to I_5. Actually he derives both (A) and (C) as consequences of a stronger (but more complicated) theorem (B). The author remarks that $(K-\operatorname{int} I_3)\times I_1$ can be shown to be homeomorphic to I_4 if and only if Poincaré's conjecture is true. For technical details of the paper see also Poenaru, Rev. Math. Pures Appl. **3** (1958), 139–156 [MR **21** #5196].

The paper uses a formalism of doubtful necessity and could be considerably reduced. The results were announced in two unreadable notes in C. R. Acad. Sci. Paris [**247** (1958), 624–626, 1818–1820; MR **21** #1601, 1602].

R. H. Fox (Princeton, N.J.)

8507:

Berstein, I. On the 1-dimensional category of Cartesian products. Proc. Cambridge Philos. Soc. **56** (1960,) 425–426.

If $P_1, \cdots, P_n$ are (2-dimensional) pseudo-projective spaces corresponding to different primes $p_1, \cdots, p_n$ then $X=P_1\times\cdots\times P_n$ can be covered by four open sets U_1, U_2, U_3, U_4 such that any loop in any U_i is contractible in X. The author states without proof that, for $n\geqq 2$, four is best possible. (For $n=1$, three is best possible.)

R. H. Fox (Princeton, N.J.)

8508:

Browder, William. Loop spaces of H-spaces. Bull. Amer. Math. Soc. **66** (1960), 316–319.

Let X be the space of loops on Y, where Y is a one-connected H-space. The author proves the following theorem: if $H^*(X)$ (the singular cohomology ring over the integers) is a finitely generated module over the integers, then X is of the same singular homotopy type as $K(G, 1)$, where G is a free abelian group. Thus the loop space of an H-space Y is infinite-dimensional unless $Y=K(G, 2)$, G free abelian.

I. M. James (Oxford)

8509:

Soltan, P. S. Dimension of inverse images in the mapping of compacta into polyhedra. Dokl. Akad. Nauk SSSR **130** (1960), 510–513 (Russian); translated as Soviet Math. Dokl. **1**, 78–81.

Given a triangulated polyhedron P, denote by $S_x(P)$ the boundary of the star of the point x, $x\in P$. For $y\in P$, let $r(y)$ designate the maximal integer r such that there is a neighborhood $U(y)$ about y such that $S_x(P)$ is aspherical in dimensions $<r$, for all $x\in U(y)$. Theorem: Let X be a compactum of finite dimension $\dim X\geqq\dim P$ and ε a positive number. Then for any map $f: X\to P$ there exists a map $g: X\to P$ such that the distance $\rho(f,g)<\varepsilon$ and that $\dim(g^{-1}(y))\leqq\dim X-r(y)-1$, for all $y\in P$. If P is an n-manifold, then $r(y)=n-1$, and the result reduces to a theorem of W. Hurewicz [S.-B. Preuss. Akad. Wiss. **24/25** (1933), 754–768]. The result is first established for polyhedral X and then extended to the general case by an approximation argument.

{Reviewer's remark: On top of p. 81 of the translation a line is missing; it should read as follows: "Let b be an arbitrary interior point of the simplex T. Then the cone $b*\Pi(T)$ is a simplicial". The translator has corrected several troublesome misprints of the original text.}

S. Mardešić (Zagreb)

8510:

Wada, Hidekazu. Note on some mapping spaces. Tôhoku Math. J. (2) **10** (1958), 143–145.

Let G_n be the space of maps of S^n into itself and let F_n be the subspace of G_n consisting of maps which leave a base point of S^n fixed. The author has previously shown that G_n is of the homotopy type of $S^n\times F_n$ if and only if $\pi_{2n+1}(S^{n+1})$ has an element of Hopf invariant one, i.e., as we now know, only in case $n=1$, 3, or 7. He now shows that for these values of n there is actually a homeomorphism between G_n and $S^n\times F_n$.

N. Stein (New York)

8511:

Terasaka, Hidetaka. On null-equivalent knots. Osaka Math. J. **11** (1959), 95–113.

Fox and Milnor have shown [Bull. Amer. Math. Soc. **63** (1957), 406] that if a knot k in 3-space is the intersection of R^3 with a locally flat polyhedral 2-sphere in 4-space R^4 then its Alexander polynomial $\Delta_k(t)$ must be of the form $f(t)f(t^{-1})$, where $f(t)$ is an integral polynomial satisfying $f(1)=1$. {For such a knot the term "null-equivalent" was used, but the reviewer feels that the choice of this term was rather poor, and hopes that a better one will be found.} The author proves that if $f(t)$ is any integral polynomial satisfying $f(1)=1$, then there is a "null-equivalent" knot with polynomial $\Delta(t)=f(t)f(t^{-1})$.

The key step in the proof of Fox and Milnor [loc. cit.] was the following theorem: Let L be a totally splittable link which consists of a knot l together with a collection of n disjoint circles. Suppose that n bands connecting the components of L are given and let k be the knot formed from L together with the bands by removing the interiors and ends of the bands. Then the Alexander polynomial $\Delta_k(t)$ has the form $\Delta_l(t)f(t)f(t^{-1})$. The author gives a new proof of this theorem for the special case $n=1$. It seems to be implied that the general case follows by induction; but this is not at all clear, since each of the n bands may loop through each of the circles.

R. H. Fox (Princeton, N.J.)

8512a:

Zeeman, E. C. Unknotting spheres in five dimensions. Bull. Amer. Math. Soc. **66** (1960), 198.

8512b:

Zeeman, E. C. Unknotting spheres. Ann. of Math. (2) **72** (1960), 350–361.

Let Δ^n denote the standard n-simplex. A combinatorial n-ball (n-sphere) is a finite simplicial complex which is piecewise linearly homeomorphic to Δ^n (to $\dot\Delta^{n+1}$). A combinatorial n-manifold M is a finite simplicial complex such that the link of each vertex is either a combinatorial $(n-1)$-sphere (if the vertex is interior to M) or a combinatorial $(n-1)$-ball (if the vertex is in the boundary $\dot M$ of M). A combinatorial n-sphere S^n embedded in E^k, $k>n$, is unknotted if there is a piecewise linear homeomorphism of E^k onto itself taking S^n onto the boundary of an $(n+1)$-simplex.

The main result of this paper is the Unknotting Theorem (theorem 2) which states that if $k > \frac{3}{2}(n+1)$, every combinatorial S^n in E^k is unknotted. Preparatory to proving this theorem it is shown (theorem 1) that the condition that S^n is unknotted in E^k is equivalent to each of several other conditions; among them are (3) S^n bounds a combinatorial $(n+1)$-ball, (4) S^n is equivalent by cellular moves to the boundary of a combinatorial $(n+1)$-ball, and (5) S^n is equivalent by simplicial moves to the boundary of an $(n+1)$-simplex.

Under the condition $k > \frac{3}{2}(n+1)$, it is then shown that for a point V in general position in E^k relative to S^n, a subdivision of S^n is the union $B_1 \cup B_2$ of combinatorial n-balls, with intersection $B_1 \cap B_2$ an S^{n-1} in the subdivision, such that each of the joins of B_1 and B_2 with V is a combinatorial $(n+1)$-ball. S^n is then equivalent by cellular moves to the boundary of a combinatorial $(n+1)$-ball.

The arguments involve techniques and results on combinatorial manifolds found by J. W. Alexander [same Ann. (2) **31** (1930), 292–320] and J. H. C. Whitehead [Proc. London Math. Soc. (2) **45** (1939), 243–327]. The method of spun knots of Artin [Abh. Math. Sem. Univ. Hamburg **4** (1926), 174–177] extends [D. B. A. Epstein, #8514 below] to show that S^n can be knotted in E^{n+2}. The author has announced [Bull. Amer. Math. Soc. **67** (1961), 117–119] that in fact S^n cannot be knotted in E^{n+i}, $i \geq 3$.

E. Dyer (Chicago, Ill.)

8513:

Zeeman, E. C. Linking spheres. Abh. Math. Sem. Univ. Hamburg **24** (1960), 149–153.

The author shows it is possible to link two tame unknotted n-spheres in Euclidean k-space if (1) $k = n+2$ or (2) $n+2 < k \leq 2n+1$ and $\pi_n(S^{k-n-1}) \neq 0$. Thus, the first unsolved case is the linking of two 10-spheres in 17-space, corresponding to the vanishing of the stable group $\pi_{n+4}(S^n)$. The case (1) is a nice application of Artin's method of spun knots. Case (2) is a direct construction in the join of S^n and S^q, where $k = n+q+1$ and $2 \leq q \leq n$.

E. Dyer (Chicago, Ill.)

8514:

Epstein, D. B. A. Linking spheres. Proc. Cambridge Philos. Soc. **56** (1960), 215–219.

Strengthening a special case of a result of E. C. Zeeman [see #8513 above], the author shows that two n-spheres can be embedded in euclidean $(n+2)$-space $(n \geq 1)$ in each of the following ways: (i) neither can be shrunk to a point in the complement of the other; (ii) one can and one cannot be shrunk to a point in the complement of the other.

E. Dyer (Chicago, Ill.)

DIFFERENTIAL TOPOLOGY

See also 8134, 8135, 8456, 8457, 8469, 8470a–b, 8495.

8515:

Palais, Richard S. Extending diffeomorphisms. Proc. Amer. Math. Soc. **11** (1960), 274–277.

Let M be a differentiable n-manifold and let ψ be a diffeomorphism between two differentially embedded k-cells in M. It is shown that ψ can be extended to be a diffeomorphism of M onto itself. If $k = n$, and M is orientable, an additional hypothesis is necessary: Either ψ must be orientation-preserving, or M must admit at least one orientation-reversing diffeomorphism onto itself.

A. M. Gleason (Cambridge, Mass.)

8516:

Weier, Joseph. Classes caractéristiques de certains champs tensoriels sur des variétés différentiables à n dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 955–959.

The author proposes a refinement of the notion of characteristic classes of a vector bundle E over a differentiable manifold V, apparently inspired by Nielsen's notion of fixed point class. Let $F_1, \cdots, F_t$ be sections of E, and let A be the "characteristic cycle", i.e., the set where the F_j are dependent, assumed to be a polyhedron of the proper dimension. A subpolyhedron B is called "characteristic component" if there exists an open connected set W with $B \subset \overline{W}$, such that the F_i can be deformed (without change on A) so as to become dependent on $A \cup \overline{W}$. Then A falls into "components" B_i, and the attached characteristic cycle is a sum of cycles z_i, attached to the B_i. The system z_i is stated to be invariant, in a certain sense, under deformation of the F_j. If a z_i is ~ 0 it may still yield a non-zero invariant through a linking process; it is then called normal; for a certain dimension range the number of normal z_i is stated to be invariant under deformation of the F_j. The invariants can be transcribed into cohomology; generalizations are mentioned.

H. Samelson (Princeton, N.J.)

8517:

Reinhart, Bruce L. The winding number on two manifolds. Ann. Inst. Fourier. Grenoble **10** (1960), 271–283.

The reviewer [Trans. Amer. Math. Soc. **87** (1958), 492–512; MR **20** #1319] has proved that on a 2-manifold M, the regular homotopy classes of regular closed curves form a group $\pi_R(M)$, naturally isomorphic to the fundamental group of the unit tangent bundle $\pi_1(T_M)$. Starting from this fact, the author shows that if M is closed and orientable, there exists one and only one homomorphism w from $\pi_R(M)$ to the integers modulo the Euler characteristic of M such that (a) w is zero on each of a standard set representing generators of $\pi_1(M)$, and (b) w has value 1 on any based positively oriented contractible regular simple closed curve. This defines a generalized winding number. For simple curves in $\pi_R(M)$, an algorithm is given for computing this winding number. This is used to prove that for a sphere and torus the winding number and homotopy class characterize the regular homotopy class.

S. Smale (Berkeley, Calif.)

8518:

Milnor, John W. Sommes de variétés différentiables et structures différentiables des sphères. Bull. Soc. Math. France **87** (1959), 439–444.

If M_1^n, M_2^n are oriented differentiable manifolds, the author defines their sum $M_1 \# M_2$ as follows. Let f_i: $D^n \to M_i$ be differentiable imbeddings of the n-disk, orientation-preserving if $i=1$, orientation-reversing if $i=2$. Then $M_1 \# M_2$ is the space $(M_1 - \text{interior } f_1(D^n)) \cup (M_2 - \text{interior } f_2(D^n))$ with points on the boundaries

identified under $f_2 f_1^{-1}$. One imposes a compatible differentiable structure on $M_1 \# M_2$. This sum is commutative and associative, with identity S^n. The author, using work of Mazur [#8469 above], studies those closed manifolds with inverses under $\#$. These form an abelian group A^n, each member of which is homeomorphic to S^n. A subgroup Γ^n of A^n is defined as those manifolds which can be obtained by pasting two n-disks together by a diffeomorphism of their boundaries. The author, summarizing known information, states that A^n is countable, $\Gamma^n = 0$ for $n \leq 4$, $\Gamma^{4k-1} \neq 0$ for $k \neq 1$, 3, and poses the question: are these groups finitely generated? The author refers to the reviewer for the "fact" that $\Gamma^4 = 0$. Unfortunately the reviewer's "proof" of this (never published) is incomplete. On the other hand, the reviewer has shown that A^n and Γ^n coincide and are finite except for possibly $n = 4$, 6, 7. The author concludes by discussing homotopy spheres under the relation of J-equivalence. In this case one obtains a group Θ^n. The author announces that Θ^n is finite for n odd, $n \neq 3$. Since this paper has appeared, the subjects treated in it have developed considerably.

S. Smale (Berkeley, Calif.)

8519:

Smale, Stephen. Morse inequalities for a dynamical system. Bull. Amer. Math. Soc. **66** (1960), 43–49.

Let M^n be a compact, n-dimensional differentiable manifold and f a real-valued function defined on it. The Morse inequalities can be interpreted as relations between the Betti numbers of M^n and the number of unstable manifolds associated to the singularities of the vector field $Z = \operatorname{grad} f$. In this important paper the author considers vector fields of a much more general type and extends to them the inequalities of Morse. Previous results in this direction were obtained by Èl'sgol'c [Mat. Sb. (N.S.) **26** (**68**) (1950), 215–223; MR **11**, 671] and Reeb [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27** (1952), no. 9; MR **15**, 336], but both make rather restrictive assumptions about the nature of the vector field. The main results of the author are contained in theorem 1 and theorem 2 below. Let X be a C^∞-vector field defined on M^n satisfying the following conditions: (1) X has only a finite number of singularities $\beta_1, \cdots, \beta_k$ and they are of simple type, i.e., for each of them the eigenvalues have non-zero real part; (2) X has only a finite number of closed orbits $\beta_{k+1}, \cdots, \beta_m$, each of simple type, i.e., for any one of them $n-1$ of the characteristic exponents have non-zero real part; (3) the α- and ω-limit sets of any orbit can only be a singular point or a closed orbit; (4) to each β_i associate the unstable manifold W_i defined as the union of all orbits γ such that $\alpha(\gamma) = \beta_i$, and similarly the stable one, W_i^*, defined as the union of all orbits γ such that $\omega(\gamma) = \beta_i$; it is then required that the stable and unstable manifolds of the β_i have normal intersection, i.e., if $x \in W_i \cap W_j^*$ and W_{ix} and W_{jx}^* are the tangent spaces of W_i and W_j^* at x, then $\dim W_i + \dim W_j^* - n = \dim(W_{ix} \cap W_{jx}^*)$; (5) if $i > k$ there is no orbit γ with $\alpha(\gamma) = \omega(\gamma) = \beta_i$. Now call a_q the number of unstable manifolds of dimension q associated to singular points and b_q the number of unstable manifolds of dimension q associated to closed orbits and put $M_q = a_q + b_q + b_{q+1}$. Theorem 1: if X satisfies conditions (1)–(5), K is any field and R_q is the rank of $H^q(M^n, K)$, then M_q and R_q satisfy Morse's relations $M_0 \geq R_0$, $M_1 - M_0 \geq R_1 - R_0$, $M_2 - M_1 + M_0 \geq R_2 - R_1 + R_0$, $\cdots$, $\sum_{k=0}^{n} (-1)^k M_k = (-1)^n \chi$, where χ is the Euler characteristic of M^n. The fact that the above theorem contains the classical theorem of Morse for a function f with non-degenerate critical points follows from theorem 2: If f is a C^∞ function on M^n with non-degenerate critical points and $X = \operatorname{grad} f$, then X can be C^1-approximated by a vector field Y which satisfies conditions (1)–(5) with no closed orbits. No proof is given of theorem 2. The fact that systems of type (1)–(5) constitute a large class of vector fields on M^n whose trajectories are well behaved seem to be at least as important as the fact that they do satisfy Morse's inequalities. The author makes some conjectures about systems of type (1)–(5), one being about the role played by them in the theory of structural stability.

M. M. Peixoto (Rio de Janeiro)

8520:

Bott, Raoul. An application of the Morse theory to the topology of Lie groups. Proc. Internat. Congress Math. 1958, pp. 423–426. Cambridge Univ. Press, New York, 1960.

This Congress note refers to a forthcoming paper. This paper has appeared and been reviewed [Ann. of Math. (2) **70** (1959), 313–337; MR **22** #987]. It fully covers the contents of the note, except perhaps for the problems stated at the end of the note.

J. F. Adams (Cambridge, England)

8521:

Reeb, Georges. Remarques sur les structures feuilletées. Bull. Soc. Math. France **87** (1959), 445–450.

The author indicates some refinements of results by himself [*Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Ind., no. 1183, Hermann, Paris, 1952; Ann. Inst. Fourier. Grenoble **6** (1955/56), 89–115; MR **14**, 1113; **18**, 407] and A. Haefliger [Comment. Math. Helv. **32** (1958), 248–329; MR **20** #6702] on foliated structures. E.g.: If in a C^∞-foliation of codimension 1 of a compact manifold V all leaves have a finite fundamental group, then $\pi_1(V)$ contains an element of infinite order. If in such a foliation the non-compact leaves have only one end and are proper, then the compact leaves form a connected set. In an analytic (real or complex) foliation of codimension >1, if a compact leaf has first Betti number 0 and trivial linear holonomy group, then there is a neighborhood such that every leaf meeting the neighborhood is contained in it and is compact and has first Betti number 0.

H. Samelson (Princeton, N.J.)

8522:

Fujimoto, Atsuo. On the structure tensor of G-structure. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 157–169.

Let X be a C^∞ manifold of dimension n and let G be a closed subgroup of $\mathrm{GL}(n, R)$, the group of $n \times n$ real, non-singular matrices. Let $\mathbf{G}$ be the Lie algebra of G. A G-structure on X is defined by a reduction of the structure group of the tangent bundle of X from $\mathrm{GL}(n, R)$ to G. Let E be the vector space of dimension n on which G acts and let $W = (E^* \wedge E^*) \otimes E$. There is a subspace $N \subset W$ invariant under G whose definition involves only the algebraic properties of the representation of $\mathbf{G}$ on E such that the structure tensor in the sense of Bernard [C. R. Acad. Sci. Paris **243** (1956), 1821–1824; MR **18**, 933] is a

cross-section of the vector bundle on X whose typical fibre is $Q=W/N$. The author discusses the relation between the torsion tensor of a Cartan connection on the principal G-bundle over X and the structure tensor. He derives sufficient conditions, involving only the algebraic properties of the representation of G on W, that (a) the torsion tensor determine the connection and (b) the group of automorphisms of the G-structure be the group of automorphisms of a Cartan connection.

R. Hermann (Belmont, Mass.)

8523:

Legrand, Gilles. ***T*-structures sur les variétés différentiables.** C. R. Acad. Sci. Paris **250** (1960), 3266–3268.

A T-structure is a sub-bundle of the complexified tangent bundle of a differentiable manifold. Several theorems (more or less special cases of general facts) are stated concerning the curvature of a connection in a T-structure, its fundamental form Ω_a^a and class (essentially the first Chern class), reduction to the special linear group, etc.; analogous facts for real T-structures.

H. Samelson (Princeton, N.J.)

8524:

Milnor, John; Spanier, Edwin. Two remarks on fiber homotopy type. Pacific J. Math. **10** (1960), 585–590.

In § 1 the authors consider the normal sphere bundle of a compact, connected, orientable manifold M^n (without boundary) differentiably embedded in euclidean space R^{n+k}. They show (theorem 1) that if k is sufficiently large, then this normal sphere bundle has the fiber homotopy type of a product bundle if and only if there is an S-map from S^n to M^n of degree one. (The precise condition on k is $k \geqq \min(n-r+2, 3)$, where $H^q(M^n)=0$ for $0<q<r$.) They obtain the following corollary. If M^n is a homology sphere, then the normal bundle of M^n in R^{n+k} has the same fiber homotopy type as a product bundle. This is a theorem of Massey [Proc. Amer. Math. Soc. **10** (1959), 959–964; MR **22**#237].

The authors' methods rely on Spanier-Whitehead duality. The key point is the observation that the Thom space of the normal bundle is S-dual to the disjoint union of M^n and a point. This has since been included in the following useful observation, which is due to Shapiro, Atiyah and perhaps others. Let α and β be orthogonal bundles over M, and let τ be the tangent bundle of M. In order that the Thom spaces M^α and M^β should be S-dual, it is sufficient that the bundle $\alpha+\beta+\tau+0_m$ should be trivial for some m, where 0_m is the trivial $\mathrm{O}(m)$-bundle.

In § 2 the authors consider the tangent sphere bundle of a compact, connected, orientable manifold M^n (without boundary); they soon assume that M^n has the homotopy type of S^n. By quoting the result on elements of Hopf invariant one, they show (theorem 2) that if the tangent bundle has the fiber homotopy type of a product bundle then $n=1$, 3 or 7 (in which case the tangent bundle actually is a product bundle).

J. F. Adams (Cambridge, England)

8525:

Ehresmann, Charles. Catégories inductives et pseudogroupes. Ann. Inst. Fourier. Grenoble **10** (1960), 307–332.

The author develops the notions and results of his papers [Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 49–77; Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 137–150; Rev. Un. Mat. Argentina **19** (1960), 48; MR **20** #2392; **22** #7148]. He introduces, by axiomatizing a pseudo-group of local transformations of a set, the notion of inductive groupoid. Given a set X, let C be the set of local maps from X into itself and S the subset of C consisting of one-to-one maps. By axiomatizing this situation, he defines an inductive category C with a groupoid S. A filtre (usually defined on a set) is defined here on a class and the class of filtres of an inductive category is studied. A groupoid [resp. an inductive category] lying above another groupoid [resp. inductive category] is defined as something which has the same kind of properties as a pseudo-group of local transformations of a spread (espace étalé) or covering space. The author concludes the paper by studying local and infinitesimal jets.

S. Kobayashi (Vancouver, B.C.)

8526:

Halder, Gita: Behari, Ram. Some properties of automorphic equivalents of vectors in Kaehler hypersurface. Proc. Nat. Inst. Sci. India. Part A **23** (1957), 405–411.

Soit V une variété kählérienne; en un point de V, soient $z^1, \cdots, z^m$ des coordonnées complexes locales. Au lieu de coordonnées réelles, on utilise les coordonnées z^j et $\bar{z}^j$; soit $g_{a\bar{b}}$ le tenseur métrique de V. On dit que deux vecteurs sont automorphiquement équivalents si leurs composantes sont des fonctions holomorphes des coordonnées et si l'un se déduit de l'autre par l'automorphisme de l'espace tangent défini par $u'^j = eu^j$ et $u'^{\bar{j}} = \bar{e}u^{\bar{j}}$, où $e = \pm i$ et où $\bar{e}$ est l'imaginaire conjugué de e. Deux vecteurs $(u^a, u^{\bar{a}})$ et $(v^a, v^{\bar{a}})$ sont dits orthogonaux si $g_{a\bar{b}}u^a v^{\bar{b}}=0$ et $g_{\bar{a}b}u^{\bar{a}}v^b=0$. Une variété analytique complexe C de dimension n plongée dans une variété kählérienne C' de dimension $n+1$ est munie de la structure kählérienne induite. On définit, par analogie avec les espaces de Riemann, le second tenseur fondamental de C et on obtient les résultats suivants: (1) Si deux vecteurs sont conjugués, en un point de C, il en est de même de leurs équivalents automorphes; (2) si n vecteurs ont des directions principales en un point de C, les n autres directions principales sont celles des équivalents automorphes des premiers; il en résulte: la courbure moyenne de C est nulle en tout point (C est une variété minimale).

P. Dolbeault (Poitiers)

8527:

Tachibana, Shun-ichi. On almost-analytic vectors in certain almost-Hermitian manifolds. Tôhoku Math. J. (2) **11** (1959), 351–363.

In an almost complex space with structure tensor $F_i{}^h$, a contravariant vector field v^h is said to be contravariant almost analytic (contrav. a. a.) when it satisfies

$$\underset{v}{\pounds}\, F_i{}^h = v^j\partial_j F_i{}^h - F_i{}^a\partial_a v^h + F_a{}^h\partial_i v^a = 0,$$

and a covariant vector field w_i is said to be covariant almost analytic (cov. a. a.) when it satisfies

$$(\partial_j F_i{}^h - \partial_i F_j{}^h)w_h - F_j{}^a\partial_a w_i + F_i{}^a\partial_j w_a = 0.$$

Now consider an almost Hermitian space with structure tensors $F_i{}^h$ and g_{ji}. If the associated form $\frac{1}{2}F_{ji}\delta\xi^j\wedge d\xi^i$ is closed, then it is necessarily coclosed, that is, the associated form is harmonic. In this case, the space is called an almost Kählerian space. An almost Hermitian space whose associated form is that of Killing is called a

K-space by the author. An example of a K-space is given by S^6.

In the paper under review, the author studies contrav. and cov. a. a. vector fields in a K-space. After a short introduction, he proves in section 1: In a compact almost Hermitian manifold M in which $\nabla^i F_i{}^h=0$, the inner product of contrav. and cov. a. a. vectors is constant over the whole M. {Reviewer's note: Since the author's lemma 1.1 is true for any compact almost complex space, this theorem is also true for any compact almost complex space.} If, in an almost Hermitian manifold M in which $\nabla^i F_i{}^h=0$, a vector w_i is closed, then $F_i{}^t w^t$ is harmonic.

In section 2, he derives some identities and proves two theorems: In a K-space, we have $K_{ji}v^j v^i \geqq K_{ji}{}^* v^j v^i$ for any vector field v^h, where K_{ji} is the Ricci tensor and $K_{ji}{}^*$ is defined by $K_{ji}{}^* = -\frac{1}{2} F_j{}^t K_{ts\sigma r} F^{sr}$. If the dimension $\geqq 4$ and a K-space is conformally flat, then the Ricci form $K_{ji}v^j v^i$ cannot be negative definite. In sections 3 and 4, he studies contrav. and cov. a. a. vectors and proves: If, in a compact K-space, a contrav. a. a. vector v satisfies $\nabla_i v^i=0$, then it is a Killing vector. If a K-space of dimension >2 is compact, a conformal Killing vector which is at the same time contrav. a. a. is a Killing vector. If, in a compact K-space, a projective Killing vector is at the same time contrav. a. a., then it is a Killing vector. In sections 5 and 6, he obtains some integral formulas and proves, by use of these, the following theorems: In a compact K-space, a necessary and sufficient condition for a vector v to be contrav. a. a. is that

$$\nabla^i\nabla_i v^h + K_i{}^h v^i = 0, \quad 2N(v)_i + (K_{ri} - K_{ri}{}^*)v^r = 0,$$

where $N(v)_h = \frac{1}{4}(\nabla^j v^i)N_{ji}{}^h$, $N_{ji}{}^h$ being the Nijenhuis tensor. In a compact K-space, a necessary and sufficient condition for a vector w to be cov. a. a. is that w_i and $\tilde{w}_i = F_i{}^a w_a$ be both harmonic. *K. Yano* (Seattle, Wash.)

8528:

Rizza, G. B. Alcune disuguaglianze per i numeri di Betti di una varietà kähleriana. Rend. Mat. e Appl. (5) **18** (1959), 414–425.

On a compact Kaehler manifold V_{2n} of real dimension $2n$, if the first Betti number $B^1=2p$ is >0, if $u^{(1)}, \cdots, u^{(p)}$ are p linearly independent holomorphic (harmonic) vector fields, and if m is the maximal rank on V_{2n} of the $(2n, p)$ matrix composed of these vectors, then the author is able to form with certain subdeterminants of the matrix sufficiently many effective harmonic tensors so as to obtain for some of the Betti numbers the following lower bounds:

$$B^k \geqq 2\sum_{i=0}^{k}\binom{m-1}{i} + (\nu-1)(2-\gamma);$$

where $k=0, 1, \cdots, m$; $\gamma=B^0$ is the number of continua composing V_{2n}; and $\nu=0$ if k is even, 1 if k is odd. The author singles out certain special cases. He also observes that, conversely, an upper bound for any one Betti number B^k gives also an upper bound for the rank number m previously introduced.

S. Bochner (Princeton, N.J.)

8529:

Rizza, Giovanni Battista. Deviazione caratteristica delle faccette piane di una varietà a struttura complessa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 662–671.

Soit V une variété analytique complexe, de dimension réelle $2n$, munie d'une métrique hermitienne définie positive ($ds^2 = g_{p\bar q}dz^p d\bar z^q$ au point de coordonnées $(z^i, \bar z^i)$, $i=1, \cdots, n$); soit E_{2n} [resp. $\mathscr{E}_n$] l'espace des vecteurs tangents [resp. vecteurs tangents complexes] à V en un point choisi une fois pour toutes. On désigne par O le vecteur nul de E_{2n}; la métrique permet de définir un produit scalaire sur les vecteurs et les bivecteurs de E_{2n}; on en déduit la notion d'angle de deux bivecteurs défini par son cosinus. On appelle facette plane (en abrégé f.p.) tout sous-espace vectoriel à deux dimensions de E_{2n} muni d'une orientation; toute f.p. est définissable par un bivecteur. Si une f.p. est l'image réelle, dans E_{2n}, du support (complexe) d'un vecteur de $\mathscr{E}_n$ orienté canoniquement, elle est dite caractéristique (f.p.c.) [voir E. Martinelli, Ann. Mat. Pura Appl. (4) **43** (1957), 313–324; MR **20** #993]. Soient E_2 une f.p., E_1 une droite de E_2 issue de O; soit $E_2(E_1)$ la f.p.c. passant par E_1, alors (théorème 1): l'angle $\delta(E_2) = (E_2, E_2(E_1))$ est indépendant de la droite E_1 et est appelé déviation caractéristique de la f.p. E_2. Si E_2 est une f.p.c., on a: $\delta(E_2)=0$. Soit $\mathfrak{J}$ l'opérateur qui définit la structure presque complexe induite par la structure analytique complexe; un repère caractéristique dans E_{2n} est constitué de n vecteurs unitaires orthogonaux R_t et de leurs transformés par $\mathfrak{J}$. Soit $E_2{}^t$ la f.p.c. définie par le bivecteur $(R_t, \mathfrak{J}(R_t))$; on a (théorème 2): $\cos\delta(E_2) = \sum_t \cos(E_2, E_2{}^t)$. Soit ω la forme extérieure $ig_{p\bar q}dz^p \wedge d\bar z^q$ associée à la métrique et soit $d\mathscr{A}(E_2)$ l'élément d'aire sur la f.p. E_2, alors (théorème 3): la restriction de ω à E_2 est égale à $d\mathscr{A}(E_2)\cos\delta(E_2)$. On déduit de cela des résultats de W. Wirtinger [Monatsh. Math. Phys. **44** (1936), 343–365]. Soient M_1, M_2 deux vecteurs; théorème 4: si le produit hermitien $g_{p\bar q}M_1{}^p M_2{}^{\bar q}$ est réel, la déviation caractéristique de la f.p. définie par M_1 et M_2 est $\pi/2$ ou $3\pi/2$. *P. Dolbeault* (Poitiers)

8530:

Rizza, G. B. Holomorphic deviation for the $2q$-dimensional sections of complex analytic manifolds. Arch. Math. **10** (1959), 170–173.

Les définitions et notations sont celles de #8529 ci-dessus. On désigne par E_r un sous-espace vectoriel, de dimension r, de E_{2n}. Un E_{2q} $(0 \leqq q \leqq n)$ est dit holomorphe quand il coïncide avec $\mathfrak{J}(E_{2q})$. Un E_q $(1 \leqq q \leqq n)$ dont tous les vecteurs sont orthogonaux à ceux de $\mathfrak{J}(E_q)$ est appelé un E_q à produit hermitien réel (en abrégé un E_q p.h.r.) parce que tout couple de vecteurs de E_q a un produit hermitien réel. Lemmes: (1) dans tout E_{2q-1}, il y a un E_q p.h.r.; (2) par tout E_q p.h.r. passe un E_{2q} holomorphe et un seul. Pour un E_{2q} arbitraire, soit $E_{2q}(E_q)$ l'espace holomorphe défini par un E_q p.h.r. de E_{2q}; alors (généralisation du théorème 1 de l'article cité): $K = \cos(E_{2q}, E_{2q}(E_q))$ est indépendant de E_q; l'angle $\delta(E_{2q})$ tel que $K=\cos\delta(E_{2q})$ est appelé la déviation holomorphe de E_{2q}. Les autres théorèmes de l'article cité sont généralisés de la même manière que le théorème 1; les démonstrations paraîtront dans un périodique italien [Rend. Accad. Naz. dei XL (4) (**10**) (1959), 191–205]. *P. Dolbeault* (Poitiers)

8531:

Levine, Harold I. A theorem on holomorphic mappings into complex projective space. Ann. of Math. (2) **71** (1960), 529–535.

Der "Erste Hauptsatz" der Theorie der meromorphen Funktionen betrifft die Verteilung der a-Stellen einer meromorphen Funktion f, deren Definitionsbereich eine "offene" Riemannsche Fläche ist. Verf. betrachtet folgende allgemeinere Situation. $f: V \to P_p(C)$ sei eine holomorphe Abbildung der nicht-kompakten komplexen Mannigfaltigkeit V in den komplexen projektiven Raum $P_p(C)$ der komplexen Dimension p ($p \geqq \dim_C V = m$). Ferner sei $D \subset V$ ein kompaktes Gebiet, dessen Rand ∂D eine reelle $(2m-1)$-dimensionale Untermannigfaltigkeit von V ist. A sei ein $(p-m)$-dimensionaler linearer Unterraum von $P_p(C)$ derart, dass $f^{-1}(A) \cap D$ eine Menge von isolierten Punkten und $f^{-1}(A) \cap \partial D$ leer ist. Mit $n(D, A)$ werde die Schnittzahl in $P_p(C)$ der singulären Ketten $f: D \to P_p(C)$ und $i: A \to P_p(C)$ (i Einbettung) bezeichnet. Schliesslich sei $v(D)$ das Volumen von $f(D)$ bezüglich der üblichen Kählermetrik von $P_p(C)$, die so normiert sei, dass jeder m-dimensionale lineare Teilraum das Volumen 1 hat.

$$v(D) = \pi^{-m} \int_D f^*(\Omega)^m,$$

wo Ω die äussere Differentialform vom Typ (1, 1) ist, die zur Kählermetrik gehört. Verf. definiert eine nur von A abhängige äussere Differentialform Λ vom Grade $2m-1$ auf $P_p(C)-A$ derart, dass für alle V, D, f gilt

$$n(D, A) - v(D) = \int_{\partial D} f^*\Lambda.$$

Man vergleiche diesen Satz mit der "integrierten Form" des ersten Hauptsatzes ($p=m=1$) bei Nevanlinna [*Eindeutige analytische Funktionen*, Springer, Berlin, 1953; MR **15**, 208]. *F. Hirzebruch* (Bonn)

8532:

Kerner, Hans. Über die Fortsetzung holomorpher Abbildungen. Arch. Math. **11** (1960), 44–49.

Ist G ein unverzweigtes Riemannsches Gebiet über einem komplexen Zahlenraum und $H(G)$ die Holomorphiehülle von G, so ist bekanntlich jede in G holomorphe Funktion zu einer in $H(G)$ holomorphen Funktion fortsetzbar. In der vorliegenden Arbeit untersucht Verf., in welchen Umfange beliebige holomorphe Abbildungen von G in die Holomorphiehülle $H(G)$ fortsetzbar sind. Er Beweist u.a.: (1) Jede holomorphe Abbildung $\tau: G \to Y$ von G in einen holomorph-vollständigen komplexen Raum Y ist zu einer holomorphen Abbildung $\bar{\tau}: H(G) \to Y$ fortsetzbar. (2) Ist G_1 ein weiteres unverzweigtes Riemannsches Gebiet und $H(G_1)$ seine Holomorphiehülle, so ist jede holomorphe Abbildung $\tau: G \to G_1$ zu einer holomorphen Abbildung $\bar{\tau}: H(G) \to H(G_1)$ fortsetzbar. (3) Wird in (2) die Abbildung τ insbesondere als eigentlich und surjektiv vorausgesetzt, so ist auch $\bar{\tau}$ eigentlich und surjektiv.

R. Remmert (Erlangen)

8533:

Kerner, Hans. Über die Automorphismengruppen kompakter komplexer Räume. Arch. Math. **11** (1960), 282–288.

This paper is devoted to a proof of the theorem that the full group G of complex analytic automorphisms of a compact complex space with singularities X [Grauert and Remmert, Math. Ann. **136** (1958), 245–318; MR **21** #2063] is a complex Lie group, with the mapping $G \times X \to X$ describing the group action complex analytic; this generalizes the well-known theorem of Bochner and Montgomery [Ann. of Math. (2) **48** (1947), 659–669; MR **9**, 174], which asserts the same conclusions for a compact complex manifold. After establishing the local compactness of G {which had been done more generally by the reviewer [J. Math. Mech. **8** (1959), 133–141; MR **21** #6446]}, the proof is generally a straightforward modification of that given by Bochner and Montgomery. Further properties of spaces with singularities are invoked only to show that the final map $G \times X \to X$ is analytic in the strong sense [Grauert and Remmert, op. cit.].

R. C. Gunning (Princeton, N.J.)

8534:

Sorani, G. Sull'indicatore logaritmico per le funzioni di più variabili complesse. Rend. Mat. e Appl. (5) **19** (1960), 130–142.

Let R denote an open set in the Euclidean space E^{2r} of the complex vectors $(z_1, \cdots, z_r)$. Let f_k, $k=1, \cdots, r$ be analytic in R. It is assumed that the sets F_k defined by $f_k=0$ have only a finite number of intersection points $O_1, \cdots, O_n$ with multiplicities $\nu_1, \cdots, \nu_n$ in R. Let $I_1, \cdots, I_n$ be disjoint neighborhoods of $O_1, \cdots, O_n$. The author proves that a set of equations $f_k = \varepsilon_k$, $k=1, \cdots, r$, where $\varepsilon_1, \cdots, \varepsilon_r$ are sufficiently small positive numbers, define an oriented r-cycle S_ν in each I_ν. The S_ν generate a subgroup, the separator subgroup, of the r-dimensional homology group of $R - \bigcup_k F_k$, and constitute a basis thereof; and an r-cycle Γ homologous to $N_1S_1 + \cdots + N_nS_n$ is said to have the separator coefficients $N_1, \cdots, N_n$. If φ is analytic in R, we have the integral formula

$$(2\pi i)^{-r} \int_\Gamma \varphi(f_1^{-1}\, df_1 \wedge \cdots \wedge f_r^{-1}\, df_r) = \sum N_h \nu_h \varphi(O_h).$$

A similar result is derived for meromorphic f_n. The results are generalizations of corresponding results for the 2-dimensional case derived by E. Martinelli [Ann. Mat. Pura Appl. (4) **39** (1955), 335–343; MR **17**, 785] and F. Severi [Atti IV Congresso Un. Mat. Ital. (Taormino, 1951), vol. I, pp. 125–140, Casa Edit. Perrella, Rome, 1953; MR **14**, 1077]. *H. Tornehave* (Copenhagen)

8535:

Scheja, Günter. Über das Auftreten von Holomorphie- und Meromorphiegebieten, die nicht holomorph-konvex sind. Math. Ann. **140** (1960), 33–50.

On sait, d'après K. Oka, qu'un domaine d'holomorphie (X, φ), étalé au-dessus de C^n, sans point de ramification est holomorphiquement convexe; il n'en est plus de même si l'on englobe dans X des points de ramification [cf. H. Grauert et R. Remmert, Math. Z. **67** (1957), 103–128; MR **19**, 317]. En substituant à la notion de domaine étalé [cf. H. Cartan, Séminaire H. Cartan de l'École Normale Supérieure, 1951–1952; MR **16**, 233] une notion de domaine "presque étalé", on obtient ici un procédé de construction de domaines (X, φ) ramifiés au-dessus de C^n et qui sont des domaines d'holomorphie non holomorphiquement convexes. Une application holomorphe φ de X dans C^n est dite presque étalée si la fibre $\varphi^{-1}[\varphi(x)]$ est un ensemble discret sur X sauf aux points d'un ensemble analytique $E \subset X$, qui sera appelé l'ensemble singulier de φ. Alors, étant donné un espace $\hat{X}$, connexe

et holomorphiquement complet, de dimension n, et une application φ presque étalée de $\hat{X}$ dans C^n, d'ensemble singulier E, il existe une fonction f holomorphe dans $X=\hat{X}-E$ qui a (X,φ) comme domaine d'holomorphie (et aussi de méromorphie). On peut choisir arbitrairement le sous-ensemble analytique $E\subset\hat{X}$ sous réserve qu'il n'ait pas de point isolé et déterminer φ, de manière que (X,φ) ait les propriétés indiquées; on utilise l'existence d'une application holomorphe φ de $\hat{X}$ dans C^n de manière que $\varphi(E)$ soit un point et que la fibre de φ soit discrète sur $\hat{X}-E$.

Si $\dim E\leq n-2$ les fonctions holomorphes sur X sont les restrictions des fonctions holomorphes sur $\hat{X}$; si $\dim E=n-1$, il existe des fonctions holomorphes sur X, singulières sur E, ce qu'on montre en remarquant que si E_r est une composante irréductible de E, de dimension $n-1$, il existe f holomorphe sur X, $f\not\equiv 0$, qui s'annule d'ordre q_r donné au moins sur E_r. *P. Lelong* (Paris)

8536:

Sato, Shawich. σ-process in complex spaces. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 196–207.

On étudié l'éclatement sur un espace analytique X relativément à un ensemble $f_0,\cdots,f_k$ de fonctions analytiques, généralisant ainsi un travail de Kreyszig [cf. Math. Ann. **128** (1955), 479–492; MR **16**, 689] dans lequel les f_i étaient les coordonnes: $f_i=0$ définit un ensemble analytique A; on considère l'application φ de X dans $X\times P^k$, où P^k est l'espace projectif de coordonnées $p_0,\cdots,p_k$; φ est défini en écrivant que le rang de la matrice des f_i, p_i est 1. L'application φ est méromorphe au sens de Remmert, son graphe X_σ' est une modification de X le long de A; X_σ étant la normalisation de X_σ', (X_σ,σ,X) est par définition l'éclatement de X relativément au système des f_i. On donne une propriété d'équivalence permettant le passage d'un système (f_i) à un système (g_j). La modification obtenue est une modification fibrée de X, la fibre étant soit un point soit un espace projectif. *P. Lelong* (Paris)

8537:

Cartan, Henri. Sur les fonctions de plusieurs variables complexes: les espaces analytiques. Proc. Internat. Congress Math. 1958, pp. 33–52. Cambridge Univ. Press, New York, 1960.

Nach einer sorgfältigen Motivierung der lokalanalytischen Mengen in komplexen Zahlenräumen wird der Begriff des komplexen Raumes eingeführt. Die Analogien zur algebraischen Geometrie treten dabei sowie bei der Erläuterung des Satzes von Cartan-Oka über die Existenz von "Normalisierungen" (dieser Aussage entspricht in der algebraischen Geometrie der Sachverhalt, daß jeder algebraischen Varietät eine "normale" algebraische Varietät "assoziiert" ist) klar hervor. Als verwandt mit algebraisch-geometrischen Fragestellungen (Eliminationstheorie) erweist sich auch ein Satz von R. Remmert über eigentliche, holomorphe Abbildungen. Der Verfasser skizziert die Beweisidee und erinnert an eine Anwendung, die eine Charakterisierung der algebraischen Struktur meromorpher Funktionenkörper auf irreduziblen, kompakten, komplexen Räumen liefert.

Sodann führt der Verfasser den Leser in Untersuchungen von ihm und von K. Stein über Quotienten komplexer Räume X nach eigentlichen Äquivalenzrelationen R ein und erörtert Anwendungen eines von ihm bewiesenen Satzes, der eine Bedingung angibt, unter der einem Quotientenraum X/R in kanonischer Weise die Struktur eines komplexen Raumes aufgeprägt werden kann. Er erhält so eine Aussage über die kompakten, komplexen Räumen zugeordneten "maximalen" projektiv-einbettbaren Quotientenräume, einen Satz, der eine scharfe Verallgemeinerung einer klassischen Aussage über Tori darstellt. So kommt der Verfasser zum wichtigen Problem der Einbettbarkeit komplexer Räume in affine oder projektive Räume. Er behandelt die starker Anwendung fähigen Aussagen von R. Remmert über holomorphe, injektive Abbildungen "holomorph-separabler" und "holomorph-vollständiger" komplexer Räume in affine Zahlenräume C^n sowie die eigenen, scharfsinnigen Überlegungen zur Struktur der Quotientenräume komplexer Räume nach eigentlich-diskontinuierlichen, diskreten Automorphismengruppen. Viel Raum wird auch Untersuchungen W. L. Bailys und I. Saterkes gewidmet; diese beschäftigen sich mit der Aufgabe, komplexe Quotientenräume, die in der Theorie der Siegelschen Modulfunktionen auftreten, biholomorph auf im Sinne der Zariski-Topologie offene Teilmengen projektivalgebraischer Mengen abzubilden.

Wesentlichen Anteil hat der Verfasser auch an den Anwendungen auf die Theorie der reell-analytischen Mannigfaltigkeiten. Hierüber berichtet er, vor allem jedoch über ein Ergebnis von H. Grauert, dem es gelang, das seit langem ungelöste allgemeine Problem zu bewältigen, eine reell-analytische Mannigfaltigkeit der reellen Dimension n reell-analytisch in einen euklidischen Zahlenraum R^{2n+1} einzubetten. Ergebnissen von H. Grauert (über die topologische und analytische Äquivalent von Faserbündeln über "holomorph-vollständigen" komplexen Räumen) ist im wesentlichen der letzte Abschnitt gewidmet.

Noch eine Reihe weiterer Ergebnisse (von M. F. Atiyah, F. Bruhat, I. Frenkel, A. Grothendieck, K. Kodaira, P. Lelong, B. Malgrange, Ch. B. Morrey, H. Whitney) werden dem Leser dargeboten. So spiegeln die Ausführungen des Verfassers die lebhafte Entwicklung der komplexen Funktionentheorie mehrerer Veränderlicher wieder. *H. Behnke* (Münster)

AUTHOR INDEX

PART A